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Homework 4 solutions

1. Write down the definitions of limit point, limit set, closure, and closed set without consulting any resources. Think about the relations of these notions. **Repeat the process again and again until you are able to write down exactly the same statements as we did in class or as written in the textbook.**

   Definitions. Let $S \subseteq \mathbb{R}$ be a subset.

   - We say $\alpha \in \mathbb{R}$ is a *limit point* of S if, for all $\epsilon > 0$, there exists $a \in S \setminus \{\alpha\}$ such that $|a - \alpha| < \epsilon$.
   - A *limit set* of S is a set of all the limit points, which is denoted by $S' = \{\alpha \in \mathbb{R} : \alpha \text{ is a limit point of } S\}$.
   - The *closure* of S is denoted by $\overline{S} := S \cup S'$, where S' is the limit set of S.
   - If $S' \subseteq S$, or equivalently $\overline{S} = S$, then we say that S is a *closed set*.

   These were taken verbatim from your professor's lecture notes. □

2. Apply Theorem 1.14 (in my lecture notes) to find the closure of the set

$$S = \left\{ \frac{1}{n} : n \in \mathbb{Z}_+ \right\}.$$

   Remark (from the professor)**.** *Note here you need to find all the limit points of S.*

   Answer. We claim that 0 is the only limit point of S; that is, if S' denotes the set of all limit points, then $S' = \{0\}$. Then the closure of S would be

$$\begin{aligned}\overline{S} &= S \cup S' \\ &= \left\{ \frac{1}{n} : n \in \mathbb{Z}_+ \right\} \cup \{0\}.\end{aligned}$$

   To show $S' = \{0\}$, we will need to show that 0 is a limit point and any other point in S is <u>not</u> a limit point. First, we will show that 0 is a limit point. Let $\epsilon > 0$ be given, and choose an integer $n_0 > \frac{1}{\epsilon}$. If $n \geq n_0$, then

$$\begin{aligned}\left| \frac{1}{n} - 0 \right| &= \frac{1}{n} \\ &\leq \frac{1}{n_0} \\ &< \epsilon.\end{aligned}$$

   So we are saying that, for any $\epsilon > 0$, there exists $\frac{1}{n} \in S \setminus \{0\}$ for some large enough n (that is, for $n \geq n_0$) such that $|\frac{1}{n} - 0| < \epsilon$. Therefore, 0 is a limit point. Next, we will show that $\frac{1}{n} \in S$ for all $n \in \mathbb{Z}_+$ is <u>not</u> a limit point of S. Suppose $m \in \mathbb{Z}_+ \setminus \{n\}$ (which implies $|m - n| \geq 1$), and choose $\epsilon = \frac{1}{mn}$. Then we have $\epsilon > 0$ and, for all $\frac{1}{m} \in S \setminus \{\frac{1}{n}\}$, we have

$$\begin{aligned}\left| \frac{1}{n} - \frac{1}{m} \right| &= \frac{|m - n|}{mn} \\ &\geq \frac{1}{mn} \\ &= \epsilon.\end{aligned}$$

   This is the negation of the definition of the limit point, which means $\frac{1}{n}$ for all $n \in \mathbb{Z}_+$ is <u>not</u> a limit point of S. Therefore, 0 is the only limit point of S. □

3. Try the following questions:

   (i) Show that a closed interval is a closed set.

   Remark. *See Example 1, part 2 of Section 1.5 (page 49) of your professor's lecture notes. As the professor said in his own remark, he did this proof in class.*

   Proof. Let $I = [a, b]$ be a closed interval in $\mathbb{R}$, and denote I' to be the set of limit points of I. Define the closure $\overline{I} := I \cup I'$ of I. To show that I is a closed set, we need to show $\overline{I} = I$, according to Definition 1.7. It suffices to show $I' \subset I$, which would imply $I \cup I' = I$, or $\overline{I} = I$. But showing $I' \subset I$ is equivalent to showing $I^c \subset (I')^c$; that is, if $x \notin I$, then $x \notin I'$. Suppose we have $x \notin I$. Then either $x < a$ or $x > b$. (In my presentation of the proof, I will prove by cases below, but you can also just say "Without loss of generality" and argue for the case $x < a$ only, because the two arguments are similar.)

- Case 1: Suppose $x < a$. Set $\epsilon = a - x$. Then $x < a$ implies $\epsilon > 0$. Consider the open interval $(x - \epsilon, x + \epsilon)$. For all $y \in (x - \epsilon, x + \epsilon)$, we have
$$\begin{aligned} y &< x + \epsilon \\ &= x + (a - x) \\ &= a, \end{aligned}$$
which implies $y \notin I$. So we conclude $(x - \epsilon, x + \epsilon) \not\subset I$, and so $(x - \epsilon, x + \epsilon) \cap I = \emptyset$. In other words, for all $x < a$, there exists $\epsilon > 0$ (because we have already set $\epsilon = a - x$) for all $z \in I \setminus \{x\}$ such that $|z - x| \geq \epsilon$. This is a direct negation of the definition of a limit point of a set I, which means y is not a limit point of I; in other words, we have $y \notin I'$. Therefore, we conclude $I^c \subset (I')^c$.
- Case 2: Suppose $x > b$. Set $\epsilon = x - b$. Then $x > b$ implies $\epsilon > 0$. Consider the open interval $(x - \epsilon, x + \epsilon)$. For all $y \in (x - \epsilon, x + \epsilon)$, we have
$$\begin{aligned} y &> x - \epsilon \\ &= x - (x - b) \\ &= b, \end{aligned}$$
which implies $y \notin I$. So we conclude $(x - \epsilon, x + \epsilon) \not\subset I$, and so $(x - \epsilon, x + \epsilon) \cap I = \emptyset$. In other words, for all $x > b$, there exists $\epsilon > 0$ (because we have already set $\epsilon = x - b$) for all $z \in I \setminus \{x\}$ such that $|z - x| \geq \epsilon$. This is a direct negation of the definition of a limit point of a set I, which means y is not a limit point of I; in other words, we have $y \notin I'$. Therefore, we conclude $I^c \subset (I')^c$.

The cases complete the proof. □

(ii) Show that the set of integers $\mathbb{Z}$ is a closed set.

Proof. Denote $\mathbb{Z}'$ to be the set of limit points of $\mathbb{Z}$. Define the closure $\overline{\mathbb{Z}} := \mathbb{Z} \cup \mathbb{Z}'$ of $\mathbb{Z}$. To show that $\mathbb{Z}$ is a closed set, we need to show $\overline{\mathbb{Z}} = \mathbb{Z}$, according to Definition 1.7. It suffices to show $\mathbb{Z}' = \emptyset$, which would imply
$$\begin{aligned} \overline{\mathbb{Z}} &= \mathbb{Z} \cup \mathbb{Z}' \\ &= \mathbb{Z} \cup \emptyset \\ &= \mathbb{Z}. \end{aligned}$$
To show $\mathbb{Z}' = \emptyset$, we need to show that no point of $\mathbb{R}$ is a limit point of $\mathbb{Z}$. If $x \in \mathbb{R}$ were a limit point of $\mathbb{Z}$, then Definition 1.6 of the professor's lecture notes states: for all $\epsilon > 0$, there exists $y \in \mathbb{Z} \setminus \{x\}$ such that $|y - x| < \epsilon$. We will need to argue by cases: if x is an integer and if x is not an integer. (Here, the argument by both cases is necessary; this time you cannot say "Without loss of generality" and prove for one case only.)

- Case 1: Suppose $x \in \mathbb{Z}$. Choose $\epsilon = \frac{1}{2}$. (In fact, you can choose any $\epsilon = a$ for all $0 < a \leq 1$.) For all $y \in \mathbb{Z} \setminus \{x\}$, which implies $|y - x| \geq 1$, we have
$$\begin{aligned} \frac{1}{2} &= \epsilon \\ &> |y - x| \\ &\geq 1, \end{aligned}$$
which is a contradiction. So $x \in \mathbb{Z}$ is not a limit point.
- Case 2: Suppose $x \in \mathbb{R} \setminus \mathbb{Z}$. Choose $\epsilon = \frac{1}{2} \min\{x - \lfloor x \rfloor, \lceil x \rceil - x\}$, where $\lfloor x \rfloor, \lceil x \rceil \in \mathbb{Z}$ are the floor and ceiling functions of x, respectively. (In fact, you can choose any $\epsilon = a$ for all $0 < a \leq \min\{x - \lfloor x \rfloor, \lceil x \rceil - x\}$.) For all $y \in \mathbb{Z} \setminus \{x\}$, which implies $|y - x| \geq \min\{x - \lfloor x \rfloor, \lceil x \rceil - x\}$, we have
$$\begin{aligned} \frac{1}{2} \min\{x - \lfloor x \rfloor, \lceil x \rceil - x\} &= \epsilon \\ &> |y - x| \\ &\geq \min\{x - \lfloor x \rfloor, \lceil x \rceil - x\}, \end{aligned}$$
which is a contradiction. So $x \in \mathbb{R} \setminus \mathbb{Z}$ is not a limit point.

Therefore, $x \in \mathbb{R}$ is not a limit point. So we conclude $\mathbb{Z}' = \emptyset$. □

(iii) Based on your proof of part (i), find an unbounded set that has no limit point.

Remark. *I am not sure what the professor means by the instructions of this part. Instead, I ended up following my own proof from part (ii) instead. Please ask your professor for clarification.*

Answer. The set $\mathbb{Z}$ is an unbounded set that has no limit point. My proof of part (ii) already established that $\mathbb{Z}$ contains no limit points. And $\mathbb{Z}$ is unbounded; otherwise, if $\mathbb{Z}$ were bounded, then there would exist $m, M \in \mathbb{R}$ such that $m \leq k \leq M$ for all $k \in \mathbb{Z}$. But $\lceil m \rceil - 1, \lceil M \rceil + 1 \in \mathbb{Z}$ are not inside the interval $m \leq k \leq M$, contradicting the definition of a bounded set. □

Remark (from the professor). *I have done part (i) in class. I put it here to make sure that you understand my proof. Make sure that eventually you can prove it without looking at my proof.*

4. Try to mimic the proof of Theorem 1.15 to do the following problem:

   Let $\{I_n\}_{n\in\mathbb{Z}_+}$ be a sequence of closed intervals, and denote by λ_n the length of I_n. Assume that it's decreasing in the sense that

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots .$$

   Show that:

   (i) $\lim\limits_{n\to\infty} \lambda_n$ exists.

   Proof. We write $I_n = [a_n, b_n]$ for all $n \in \mathbb{Z}_+$. Since we have $I_n \supseteq I_{n+1}$, or in other words $[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$, we have $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$. So we obtain in particular two results:

   - $\{a_n\}_{n=1}^{\infty}$ is a monotone increasing sequence that is bounded above by b_1 (or by any b_n). By Theorem 1.11 of the professor's lecture notes (Monotone Convergence Theorem), we conclude that $\{a_n\}_{n=1}^{\infty}$ is convergent, and so we can set $\lim\limits_{n\to\infty} a_n = \xi$.
   - $\{b_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence that is bounded below by a_1 (or by any a_n). By Theorem 1.11 of the professor's lecture notes (Monotone Convergence Theorem), we conclude that $\{b_n\}_{n=1}^{\infty}$ is convergent, and so we can set $\lim\limits_{n\to\infty} b_n = \eta$.

   Now, as λ_n denotes the length of I_n, we can write $\lambda_n = b_n - a_n$. So, by the Sum Law, we have

$$\begin{aligned}\lim_{n\to\infty} \lambda_n &= \lim_{n\to\infty} (b_n - a_n)\\ &= \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n\\ &= \eta - \xi,\end{aligned}$$

   meaning $\lim\limits_{n\to\infty} \lambda_n$ exists. □

   (ii) if $\lim\limits_{n\to\infty} \lambda_n > 0$, then the intersection of those intervals, $\bigcap\limits_{n\in\mathbb{Z}_+} I_n$, is a closed interval with length $\lim\limits_{n\to\infty} \lambda_n$.

   Proof. Since we have $a_n \leq b_n$ for all $n \in \mathbb{Z}_+$, by Theorem 1.10 of the professor's lecture notes, we obtain $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$, or equivalently $\xi \leq \eta$. So we have

$$\begin{aligned}\lim_{n\to\infty} \lambda_n &= \eta - \xi\\ &\geq 0.\end{aligned}$$

   However, as we are now assuming $\lim\limits_{n\to\infty} \lambda_n > 0$, we must now have $\xi \neq \eta$. Indeed, if $\xi = \eta$, then we would have

$$\begin{aligned}0 &= \eta - \xi\\ &= \lim_{n\to\infty} \lambda_n\\ &> 0,\end{aligned}$$

   which is a contradiction. With $\xi \neq \eta$, we now conclude the strict inequality $\xi < \eta$, which implies that $[\xi, \eta]$ is a closed interval with length $\lim\limits_{n\to\infty} \lambda_n = \eta - \xi$. Now, as $\{a_n\}_{n=1}^{\infty}$ is a monotone increasing sequence, we have $a_n \leq \xi$ for all $n \in \mathbb{Z}_+$. Likewise, as $\{b_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence, we have $b_n \geq \eta$ for all $n \in \mathbb{Z}_+$. Altogether, we have

$$\begin{aligned}a_n &\leq \xi\\ &< \eta\\ &\leq b_n\end{aligned}$$

   for all $n \in \mathbb{Z}_+$, which implies $[\xi, \eta] \subseteq [a_n, b_n] = I_n$ for all $n \in \mathbb{Z}_+$. This implies $\bigcap\limits_{n\in\mathbb{Z}_+} I_n = [\xi, \eta]$, which indeed has length $\lim\limits_{n\to\infty} \lambda_n = \eta - \xi$. □