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Homework 5 solutions

1. Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a function. Write down the definitions of $\lim_{x\to\xi} f(x) = A$, $\lim_{x\to\xi+} f(x) = A$, $\lim_{x\to\xi-} f(x) = A$, and the definition of f *is continuous at* ξ. What does it mean by saying f *is continous on* I? **Repeat the process again and again until you can easily write down exactly the same statements as we did in class or as written in the textbook.**

   Definitions. We state the following definitions.

   - We say $\lim_{x\to\xi} f(x) = A$ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - A| < \epsilon$ for all $x \in I \setminus \{\xi\}$ satisfying $|x - \xi| < \delta$ (that is, for all $x \in (\xi - \delta, \xi + \delta)$).
   - We say $\lim_{x\to\xi^-} f(x) = A$ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - A| < \epsilon$ for all $x \in (\xi - \delta, \xi)$.
   - We say $\lim_{x\to\xi^+} f(x) = A$ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - A| < \epsilon$ for all $x \in (\xi, \xi + \delta)$.
   - We say that f is *continuous at* ξ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(\xi)| < \epsilon$ for all $|x - \xi| < \delta$ and for all $x \in I$ (or equivalently, for all $x \in (\xi - \delta, \xi + \delta) \cap I$).

   These were taken verbatim from your professor's lecture notes. □

2. Show by definition that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$, is continuous on $\mathbb{R}$.

   Hint: You may need to use the triangle inequality.

   Proof. Let $\epsilon > 0$ be given, and choose $\delta = \epsilon$. For all $x \in (\xi - \delta, \xi + \delta)$ (or if $|x - \xi| < \delta$), we can use the reverse triangle inequality to obtain

   $$\begin{aligned} |f(x) - f(\xi)| &= ||x| - |\xi|| \\ &\le |x - \xi| \\ &< \delta \\ &= \epsilon. \end{aligned}$$

   Therefore, f is continuous on $\mathbb{R}$. □

3. Define $f : \mathbb{R} \to \mathbb{R}$ to be

   $$f(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0. \end{cases}$$

   (1) Show by definition that f is continuous at every $x \neq 0$.

   Proof. We will prove by cases that f is continuous at ξ if $\xi \neq 0$. (Or just prove for the case $x > 0$ and say "Without loss of generality".)

   - Case 1: Suppose $\xi > 0$. Let $\epsilon > 0$ be given, and choose $\delta = \min\{\frac{\xi}{2}, \epsilon\}$, which implies $(\xi - \delta, \xi + \delta) \subset (0, \infty)$ and $\delta \le \epsilon$. For all $x \in (\xi - \delta, \xi + \delta)$ (or if $|x - \xi| < \delta$), we have

     $$\begin{aligned} |f(x) - f(\xi)| &= |1 - 1| \\ &= 0 \\ &< \delta \\ &\le \epsilon. \end{aligned}$$

     So f is continuous at any $\xi > 0$.
   - Case 2: Suppose $\xi < 0$. Let $\epsilon > 0$ be given, and choose $\delta = \min\{-\frac{\xi}{2}, \epsilon\}$, which implies $(\xi - \delta, \xi + \delta) \subset (-\infty, 0)$ and $\delta \le \epsilon$. For all $x \in (\xi - \delta, \xi + \delta)$ (or if $|x - \xi| < \delta$), we have

     $$\begin{aligned} |f(x) - f(\xi)| &= |(-1) - (-1)| \\ &= 0 \\ &< \delta \\ &\le \epsilon. \end{aligned}$$

     So f is continuous at any $\xi > 0$.

Therefore, f is continuous at every $\xi \neq 0$ (or $x \neq 0$). $\square$

Remark. *For the case $\xi > 0$ (the case $\xi < 0$ is similar), I chose $\delta = \min\{\frac{\xi}{2}, \epsilon\}$ in my proof, whereas the professor chose simply $\delta = \frac{\xi}{2}$. Both choices of δ work here because f is constant on the separate domains $[0, \infty)$ and $(-\infty, 0)$. However, note that the choice $\delta = \frac{\xi}{2}$ does not work for functions that are non-constant over a domain, such as $g : (0, \infty) \to \mathbb{R}$ defined by $g(x) = \frac{1}{x}$ on the domain $(0, \infty)$ (c.f. Exercise 3 in Homework 6).*

(2) Compute by definition $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$.

Proof. First, we claim $\lim_{x \to 0^+} f(x) = 1$. Let $\epsilon > 0$ be given, and choose $\delta = \min\{\frac{\xi}{2}, \epsilon\}$, which implies $(\xi, \xi + \delta) \subset (-\infty, 0)$ and $\delta \leq \epsilon$. For all $x \in (\xi, \xi + \delta)$, we have

$$\begin{aligned} |f(x) - 1| &= |1 - 1| \\ &= 0 \\ &< \delta \\ &\leq \epsilon. \end{aligned}$$

This proves $\lim_{x \to 0^+} f(x) = 1$.

Next, we claim $\lim_{x \to 0^-} f(x) = -1$. Let $\epsilon > 0$ be given, and choose $\delta = \min\{-\frac{\xi}{2}, \epsilon\}$, which implies $(\xi - \delta, \xi) \subset (-\infty, 0)$ and $\delta \leq \epsilon$. For all $x \in (\xi - \delta, \xi)$, we have

$$\begin{aligned} |f(x) - (-1)| &= |(-1) - (-1)| \\ &= 0 \\ &< \delta \\ &\leq \epsilon. \end{aligned}$$

This proves $\lim_{x \to 0^-} f(x) = -1$. $\square$

(3) Show that f is discontinuous at $x = 0$.
Hint: You may need to follow the proof of continuity of constant functions.

Proof. According to part (2), we have

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= 1, \\ \lim_{x \to 0^-} f(x) &= -1. \end{aligned}$$

The left-sided and right-sided limits are different. So f is discontinuous at $\xi = 0$. $\square$