MATH 150A discussion — Ryan Ta
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Homework 6 solutions

1. Show by defintion that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)$, is continuous on $\mathbb{R}$.

   Hint: You may try to mimic the proof of conitinuity of $f(x) = \sin(x)$ on $\mathbb{R}$. You may need to use the formula

$$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right).$$

   Proof. We would like to show that f is continuous at any $\xi \in \mathbb{R}$. From Section 2.1 of the professor's lecture notes (bottom of page 70 and bottom of page 71), we have

$$\begin{aligned}\cos a - \cos b &= -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right),\\ |\sin a| &\le |a|,\\ |\sin a| &\le 1\end{aligned}$$

   for all $a, b \in \mathbb{R}$. Now, let $\epsilon > 0$ be given, and choose $\delta = \epsilon$. For all $x \in (\xi - \delta, \xi + \delta)$ (or if $|x - \xi| < \delta$), we have

$$\begin{aligned}|f(x) - f(\xi)| &= |\cos x - \cos\xi|\\ &= \left|-2\sin\left(\frac{x+\xi}{2}\right)\sin\left(\frac{x-\xi}{2}\right)\right|\\ &= 2\left|\sin\left(\frac{x+\xi}{2}\right)\right|\left|\sin\left(\frac{x-\xi}{2}\right)\right|\\ &\le 2\cdot 1\left|\sin\left(\frac{x-\xi}{2}\right)\right|\\ &= 2\left|\sin\left(\frac{x-\xi}{2}\right)\right|\\ &\le 2\left|\frac{x-\xi}{2}\right|\\ &= |x-\xi|\\ &< \delta\\ &= \epsilon.\end{aligned}$$

   Therefore, f is continuous on $\mathbb{R}$. □

2. Show by definition that the function $f : [0, \infty) \to \mathbb{R}$ where $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

   Hint: You have to consider two different cases: (i) continuity at $\xi > 0$; (ii) continuity at $\xi = 0$. Note in the second case, one can only approach $\xi = 0$ from right, i.e. case (ii) is basically $\lim_{x\to 0^+} f(x) = f(0)$.

   Proof. We will consider two cases: if $\xi = 0$ and if $\xi > 0$.

   - Case 1: Suppose $\xi = 0$. Let $\epsilon > 0$ be given, and choose $\delta = \epsilon^2$. For all $x \in [0, 0 + \delta)$ (or if $0 \le x < \delta$), we have

$$\begin{aligned}|f(x) - f(0)| &= |\sqrt{x} - \sqrt{0}|\\ &= \sqrt{x}\\ &< \sqrt{\delta}\\ &= \sqrt{\epsilon^2}\\ &= \epsilon.\end{aligned}$$

     So f is continuous at $\xi = 0$.
   - Case 2: Suppose $\xi > 0$. Let $\epsilon > 0$ be given, and choose $\delta = \min\{\frac{\xi}{2}, \sqrt{\xi}\epsilon\}$, which implies $(\xi - \delta, \xi + \delta) \subset [0, \infty)$ and

$\delta \leq \sqrt{\xi}\epsilon$). For all $x \in (\xi - \delta, \xi + \delta)$ (or if $|x - \xi| < \delta$), we have

$$\begin{aligned}
|f(x) - f(\xi)| &= |\sqrt{x} - \sqrt{\xi}| \\
&= |\sqrt{x} - \sqrt{\xi}| \frac{|\sqrt{x} + \sqrt{\xi}|}{|\sqrt{x} + \sqrt{\xi}|} \\
&= \frac{|x - \xi|}{|\sqrt{x} + \sqrt{\xi}|} \\
&= \frac{|x - \xi|}{\sqrt{x} + \sqrt{\xi}} \\
&\leq \frac{|x - \xi|}{\sqrt{\xi}} \\
&< \frac{\delta}{\sqrt{\xi}} \\
&\leq \frac{\sqrt{\xi}\epsilon}{\sqrt{\xi}} \\
&= \epsilon.
\end{aligned}$$

So f is continuous at every $\xi > 0$.

Therefore, f is continuous on $[0, \infty)$. □

Remark. *For Case 2, it is slightly imprecise to choose* $\delta = \sqrt{\xi}\epsilon$*; one should indeed choose* $\delta = \min\{\frac{\xi}{2}, \sqrt{\xi}\epsilon\}$ *because you want to preclude any possibility of constructing an interval* $(\xi - \delta, \xi + \delta)$ *centered at* $\xi > 0$ *that is* <u>*not*</u> *contained in* $[0, \infty)$*, the domain of* f *given by* $f(x) = \sqrt{x}$*. However, this is a subtle issue that you do not need to be too concerned about in this class; the professor and I will give full credit even if you just wrote* $\delta = \sqrt{\xi}\epsilon$*.*

3. Consider the function $f(x) : (0, \infty) \to \mathbb{R}$ where $f(x) = \frac{1}{x}$.

   (a) Show by definition that f is continuous at every $\xi > 0$.

   Proof. Suppose $\xi > 0$. Let $\epsilon > 0$ be given, and choose $\delta = \min\{\frac{\xi}{2}, \frac{\xi^2}{2}\epsilon\}$, which implies $\delta \leq \frac{\xi}{2}$ and $\delta \leq \frac{\xi^2}{2}\epsilon$. For all $x \in (\xi - \delta, \xi + \delta)$ (or if $|x - \xi| < \delta$), we get $x > \xi - \delta \geq \xi - \frac{\xi}{2} = \frac{\xi}{2}$, and so we have

$$\begin{aligned}
|f(x) - f(\xi)| &= \left|\frac{1}{x} - \frac{1}{\xi}\right| \\
&= \left|\frac{\xi}{x\xi} - \frac{x}{x\xi}\right| \\
&= \frac{|x - \xi|}{x\xi} \\
&< \frac{|x - \xi|}{(\frac{\xi}{2})\xi} \\
&< \frac{\delta}{(\frac{\xi}{2})\xi} \\
&= \frac{2}{\xi^2}\delta \\
&\leq \frac{2}{\xi^2}\left(\frac{\xi^2}{2}\epsilon\right) \\
&= \epsilon.
\end{aligned}$$

   So f is continuous at any $\xi > 0$. □

   (b) Show that f does not have a right-sided limit at $\xi = 0$. Here you may use Theorem 2.2 and its corollaries.

   Proof. Suppose instead there exists $A \in \mathbb{R}$ such that $\lim_{x \to 0^+} f(x) = A$. By Theorem 2.2, if $\{x_n\}_{n=1}^{\infty}$ is a sequence that satisfies $x_n > 0$ for all $n \in \mathbb{Z}_+$ and $\lim_{n\to\infty} x_n = 0$, then we also have $\lim_{n\to\infty} f(x_n) = A$. Let $x_n = \frac{1}{n}$ for all $n \in \mathbb{Z}_+$, then we would have

$$\begin{aligned}
\lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{1}{n} \\
&= 0.
\end{aligned}$$

We also have

$$\begin{aligned} f(x_n) &= \frac{1}{x_n} \\ &= \frac{1}{\frac{1}{n}} \\ &= n, \end{aligned}$$

which means $\{f(x_n)\}_{n=1}^{\infty}$ is an unbounded sequence. Therefore, $\{f(x_n)\}_{n=1}^{\infty}$ is a divergent sequence, and so the right-sided limit $\lim_{x \to 0^+} f(x)$ does not exist. □

4. Prove Theorem 2.4 via Theorem 2.2.

Statement of Theorem 2.2 (page 73 of the professor's lecture notes)**:** Let $f : I \to \mathbb{R}$ be a function. Let $\xi \in I$. Then f is continuous at ξ if and only if the following statement is true: For all $\{x_n\}_{n=1}^{\infty}$ on I with $\lim_{n\to\infty} x_n = \xi$, $\{f(x_n)\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} f(x_n) = f(\xi)$.

Statement of Theorem 2.4 (page 79 of the professor's lecture notes)**:** If $f, g : I \to \mathbb{R}$ are continuous at $\xi \in I$, then fg is continuous at I.

Proof. Since f, g are continuous at ξ, by Theorem 2.2, the following statement is true: For all $\{x_n\}_{n=1}^{\infty}$ on I with $\lim_{n\to\infty} x_n = \xi$, $\{f(x_n)\}_{n=1}^{\infty}, \{g(x_n)\}_{n=1}^{\infty}$ are convergent, $\lim_{n\to\infty} f(x_n) = f(\xi)$, and $\lim_{n\to\infty} g(x_n) = g(\xi)$. By the Product Law, $\{fg(x_n)\}_{n=1}^{\infty}$ is also a convergent sequence that satisfies

$$\lim_{n\to\infty}(f(x_n)g(x_n)) = (\lim_{n\to\infty} f(x_n))(\lim_{n\to\infty} g(x_n)).$$

In fact, we have

$$\begin{aligned} \lim_{n\to\infty}(fg)(x_n) &= \lim_{n\to\infty}(f(x_n)g(x_n)) \\ &= (\lim_{n\to\infty} f(x_n))(\lim_{n\to\infty} g(x_n)) \\ &= f(\xi)g(\xi) \\ &= (fg)(\xi). \end{aligned}$$

By Theorem 2.2, fg is continuous at ξ. □