MATH 150A discussion
University of California, Riverside

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Homework 7 solutions

Reminder: If you choose to view these solutions, please only do so whenever you cannot make further progress on solving a problem. Make sure you understand firmly what is being written, and make sure that you are writing your own proof in your own words!

1. Compute
$$\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x}.$$
Hint: Note this function is not defined at $x = 0$. But you may try to convert it into a function that is continuous on $\mathbb{R}$. Then use that function to compute the limit.

Proof. We can multiply and divide by $\sqrt{x^2+1}+1$, which is the conjugate of $\sqrt{x^2+1}-1$, to obtain
$$\begin{aligned}\frac{\sqrt{x^2+1}-1}{x} &= \frac{\sqrt{x^2+1}-1}{x}\frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1} \\ &= \frac{(\sqrt{x^2+1})^2-1^2}{x(\sqrt{x^2+1}+1)} \\ &= \frac{(x^2+1)-1}{x(\sqrt{x^2+1}+1)} \\ &= \frac{x^2}{x(\sqrt{x^2+1}+1)} \\ &= \frac{x}{\sqrt{x^2+1}+1}.\end{aligned}$$
Therefore, the limit is
$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{x} &= \lim_{x\to 0}\frac{x}{\sqrt{x^2+1}+1} \\ &= \frac{0}{\sqrt{0^2+1}+1} \\ &= 0,\end{aligned}$$
as desired. □

2. (a) Prove that if $f : I \to \mathbb{R}$ is continuous on I, then $|f| : I \to \mathbb{R}$ where
$$|f|(x) := |f(x)|$$
is continuous for all $x \in I$.

Proof. Define $g : I \to \mathbb{R}$ by $g(x) := |x|$. Then g is continuous on I, by Exercise 2 of Homework 5. Now, we can write
$$\begin{aligned}|f|(x) &= |f(x)| \\ &= g \circ f(x)\end{aligned}$$
for all $x \in I$. In other words, we have $|f| = g \circ f$, which is the composition of two continuous functions f, g on I. By Theorem 2.6 (pages 82-83) of the professor's lecture notes, the composition of continuous functions is a continuous function; in other words, $|f|$ is continuous on I. □

(b) Find a function f such that $|f|$ is continuous at some $\xi \in I$ and f is discontinuous at ξ.

Proof. Let $I = \mathbb{R}$ and $\xi = 0$. Define $f : I \to \mathbb{R}$ by, for example,
$$f(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$
Then f is discontinuous at $\xi = 0$, according to Exercise 3, part (1) of Homework 5. However, we also get
$$\begin{aligned}|f|(x) &= |f(x)| \\ &= \begin{cases} |1| & \text{if } x \geq 0, \\ |-1| & \text{if } x < 0 \end{cases} \\ &= \begin{cases} 1 & \text{if } x \geq 0, \\ 1 & \text{if } x < 0 \end{cases} \\ &= 1.\end{aligned}$$

In other words, $|f|$ is a constant function (equivalently, a power function with the zero exponent, such as x^0) on $I = \mathbb{R}$. As a power function, $|f|$ is continuous on $I = \mathbb{R}$, according to page 65 of the professor's lecture notes. In particular, $|f|$ is continuous at $\xi = 0$. □

3. (a) Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \cos\left(\sqrt{x^2 + \left(\frac{\pi}{2}\right)^2}\right)$$

is continuous on $\mathbb{R}$.

Proof. Define $h_1, h_2 : \mathbb{R} \to [0, \infty)$ by

$$h_1(x) := x^2,$$
$$h_2(x) := \left(\frac{\pi}{2}\right)^2.$$

Then h_1 is continuous, according to pages 68-69 of your professors lecture notes, and h_2 is continuous because it is a constant function. Furthermore, define $g_1 : \mathbb{R} \to [0, \infty)$, $g_2 : [0, \infty) \to \mathbb{R}$, and $g_3 : \mathbb{R} \to \mathbb{R}$ by

$$g_1(x) := h_1(x) + h_2(x),$$
$$g_2(x) := \sqrt{x},$$
$$g_3(x) := \cos x.$$

We note that g_1 is the addition of the two continuous functions h_1, h_2 on $\mathbb{R}$. By Theorem 2.3 (page 77) of the professor's lecture notes, the sum of any two continuous functions is a continuous function; in other words, g_1 is continuous on $\mathbb{R}$. Additionally, g_2, g_3 are continuous by Exercises 2 and 1 of Homework 6, respectively. Finally, we can write f as

$$\begin{aligned} f(x) &= \cos\left(\sqrt{x^2 + \left(\frac{\pi}{2}\right)^2}\right) \\ &= \cos\left(\sqrt{h_1(x) + h_2(x)}\right) \\ &= \cos\left(\sqrt{g_1(x)}\right) \\ &= \cos(g_2 \circ g_1(x)) \\ &= g_3 \circ g_2 \circ g_1(x). \end{aligned}$$

In other words, $f = g_3 \circ g_2 \circ g_1$ is a composition of the continuous functions g_1, g_2, g_3. By Theorem 2.6 (pages 82-83) of the professor's lecture notes, the composition of continuous functions is a continuous function; in other words, f is continuous on $\mathbb{R}$. □

Remark. *In my proof above, I denoted g_1, g_2, g_3, h_1, h_2 to be the continuous functions that one has to compose to obtain f. But you can choose your own notation for such functions. For instance, you can instead call these five continuous functions g, h, i, j, k. We only have to use some kind of notation, and for me personally I chose g_1, g_2, g_3, h_1, h_2.*

(b) Compute

$$\lim_{n\to\infty} \cos\left(\sqrt{\frac{1}{n^2} + \left(\frac{\pi}{2}\right)^2}\right).$$

Answer. Define the sequence $\{x_n\}_{n=1}^{\infty}$ by $x_n := \frac{1}{n}$. Then we have $\lim_{n\to\infty} x_n = 0$. According to part (a), f is continuous. By Theorem 2.2 (page 73 of the professor's lecture notes), we have $\lim_{n\to\infty} f(x_n) = f(0)$. In fact, we obtain

$$\begin{aligned} \lim_{n\to\infty} \cos\left(\sqrt{\frac{1}{n^2} + \left(\frac{\pi}{2}\right)^2}\right) &= \lim_{n\to\infty} \cos\left(\sqrt{(x_n)^2 + \left(\frac{\pi}{2}\right)^2}\right) \\ &= \lim_{n\to\infty} f(x_n) \\ &= f(0) \\ &= \cos\left(\sqrt{(0)^2 + \left(\frac{\pi}{2}\right)^2}\right) \\ &= \cos\left(\frac{\pi}{2}\right) \\ &= 0, \end{aligned}$$

as desired. □

4. Construct following three types of functions.

(1) $f : I \longrightarrow \mathbb{R}$ is continuous and I is bounded. But f is unbounded.

Answer. Let $I = (0, 1]$ and define $f : I \longrightarrow \mathbb{R}$ by $f(x) = \frac{1}{x}$. Then I is bounded because we have $0 < x \leq 1$ for all $x \in I$. And f is unbounded on I because, if we consider a sequence $\{x_n\}_{n=1}^{\infty}$ in I given by $x_n = \frac{1}{n}$, then we have $f(x_n) = f(\frac{1}{n}) = \frac{1}{\frac{1}{n}} = n$. And we have already proved in Exercise 3, part (a) of Homework 6 that f is continuous on I. □

(2) $f : I \longrightarrow \mathbb{R}$ where I is closed and bounded. But f is unbounded.

Answer. Let $I = [0, 1]$ and define $f : I \longrightarrow \mathbb{R}$ by

$$f(x) := \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{x} & \text{if } 0 < x \leq 1. \end{cases}$$

Then I is closed because we have $\overline{I} = \overline{[0, 1]} = [0, 1] = I$. And f is unbounded on I for the same reasons as given in part (a). □

(3) $f : I \longrightarrow \mathbb{R}$ is continuous and I is bounded. And f has a minimum point but doesn't have a maximum point.

Answer. Let $I = (0, 1]$ and define $f : I \longrightarrow \mathbb{R}$ by $f(x) = \frac{1}{x}$. Then I is bounded because we have $0 < x \leq 1$ for all $x \in I$. And f has a minimum point on I because $f(x) \geq 1$ for all $x \in I$. But f does not have a maximum because it is unbounded. And we have already proved in Exercise 3, part (a) of Homework 6 that f is continuous on I. □

Remark. *These are my choices of functions and intervals that satisfy the required conditions. But there are infinitely many choices of functions and intervals satisfying the required conditions. Please construct your own examples.*