HOMEWORK ASSIGNMENT EIGHT

MATH 150A, WINTER 2020

1. Let f be a continuous function defined on a closed, bounded interval $I = [a, b]$. Suppose that f assume its maximum M at some $\xi \in (a, b)$. Show that f cannot be one-to-one. Similarly, if f assume its minimum m at some $\xi \in (a, b)$, then f cannot be one-to-one.

Hint: you need to use the Intermediate Value Theorem.

2. Let f be a continuous function defined on a closed, bounded interval $I = [a, b]$. Assume that f is one-to-one. Let m (M, respectively) be the minimum (maximum, respectively) of f. Then by problem 1, we know that either $f(a) = m$ and $f(b) = M$; or $f(a) = M$ and $f(b) = m$. If $f(a) = m$ and $f(b) = M$, then show that f is strictly monotone increasing. If $f(a) = M$ and $f(a) = m$, then show that f is strictly monotone decreasing.

Hint: you need to use the Intermediate Value Theorem and argue by contradition.

Remark: Problem 2 basically says the only way for a continuous function to be one-to-one is to be strictly monotone. In other words, only strictly monotone continuous functions can have inverse. This is actually a Theorem in our text.

3. Consider $f : I \to \mathbb{R}$. Let J be an interval containing $f(I)$. Let $g : J \to \mathbb{R}$. Consider the composition of f and g, i.e. $g \circ f : I \to \mathbb{R}$. Show that:

(1) If both f and g are strictly monotone increasing, then $g \circ f : I \to \mathbb{R}$ is strictly monotone increasing.
(2) If both f and g are strictly monotone decreasing, then $g \circ f : I \to \mathbb{R}$ is strictly monotone increasing.
(3) If one of f and g is strictly monotone increasing and the other is strictly monotone decreasing, then $g \circ f : I \to \mathbb{R}$ is strictly monotone decreasing.

4. Let $f : \mathbb{R} \to \mathbb{R}$ to be a contiuous, strictly monotone function. Let $f(\mathbb{R}) = \mathbb{R}$. Thus its the inverse f^{-1} is defined on the whole real line $\mathbb{R}$. Show that $f^{-1} : \mathbb{R} \to \mathbb{R}$ continous on $\mathbb{R}$.

Hint: note Theorem 2.10 only deals with the case where the domain is a closed, bounded interval. So you cannot apply it directly to problem 4. But you may follow the proof of continuity of $g(x) = \sqrt{x} : [0, \infty) \to [0, \infty)$, which was considered as the inverse of $f(x) = x^2 : [0, \infty) \to [0, \infty)$.