 Ryan Ta

Homework 8 solutions

Reminder: If you choose to view these solutions, please only do so whenever you cannot make further progress on solving a problem. Make sure you understand firmly what is being written, and make sure that you are writing your own proof in your own words!

1. Let f be a continuous function defined on a closed, bounded interval $I = [a, b]$. Suppose that f assumes its maximum M at some $\xi \in (a, b)$. Show that f cannot be one-to-one. Similarly, if f assumes its minimum m at some $\xi \in (a, b)$, then f cannot be one-to-one.

   Hint: You need to use the Intermediate Value Theorem.

   Proof. First, suppose f assumes its maximum M at $\xi \in (a, b)$. Then we have $f(x) \leq f(\xi)$ for all $x \in I = [a, b]$, prompting us to consider the following cases.

   - Case 1: Suppose $f(a) = f(\xi)$. As (a, b) is an open interval, note that $\xi \in (a, b)$ implies the inequality $a < \xi$, which in turn implies $a \neq \xi$. But we also have $f(a) = f(\xi)$ by our assumption for this case. Therefore, f is not one-to-one.
   - Case 2: Suppose $f(b) = f(\xi)$. As (a, b) is an open interval, note that $\xi \in (a, b)$ implies the inequality $\xi < b$, which in turn implies $\xi \neq b$. But we also have $f(\xi) = f(b)$ by our assumption for this case. Therefore, f is not one-to-one.
   - Case 3: Suppose $f(a) < f(\xi)$ and $f(b) < f(\xi)$. Let γ be some number such that $\max\{f(a), f(b)\} < \gamma < f(\xi)$. By the Intermediate Value Theorem, there exists $c_1 \in (a, \xi) \subset I$ such that $f(c_1) = \gamma$. Also by the Intermediate Value Theorem, there exists $c_2 \in (\xi, b) \subset I$ such that $f(c_2) = \gamma$. Note that $c_1 < \xi < c_2$ implies $c_1 \neq c_2$. But we also have $f(c_1) = \gamma = f(c_2)$. Therefore, f is not one-to-one.

   In all three cases, we have proved that f is not one-to-one.

   Next, suppose f assumes its minimum m at $\xi \in (a, b)$. Then we have $f(x) \geq f(\xi)$ for all $x \in I = [a, b]$, prompting us to consider the following cases.

   - Case 1: Suppose $f(a) = f(\xi)$. As (a, b) is an open interval, note that $\xi \in (a, b)$ implies the inequality $a < \xi$, which in turn implies $a \neq \xi$. But we also have $f(a) = f(\xi)$ by our assumption for this case. Therefore, f is not one-to-one.
   - Case 2: Suppose $f(b) = f(\xi)$. As (a, b) is an open interval, note that $\xi \in (a, b)$ implies the inequality $\xi < b$, which in turn implies $\xi \neq b$. But we also have $f(\xi) = f(b)$ by our assumption for this case. Therefore, f is not one-to-one.
   - Case 3: Suppose $f(a) > f(\xi)$ and $f(b) > f(\xi)$. Let γ be some number such that $\min\{f(a), f(b)\} > \gamma > f(\xi)$. By the Intermediate Value Theorem, there exists $c_1 \in (a, \xi) \subset I$ such that $f(c_1) = \gamma$. Also by the Intermediate Value Theorem, there exists $c_2 \in (\xi, b) \subset I$ such that $f(c_2) = \gamma$. Note that $c_1 < \xi < c_2$ implies $c_1 \neq c_2$. But we also have $f(c_1) = \gamma = f(c_2)$. Therefore, f is not one-to-one.

   In all three cases, we have proved that f is not one-to-one. □

   Remark. *The reason we needed to break our argument into cases is that, given the interval $I = [a, b]$, the Intermediate Value Theorem can only be used for continuous functions $f : I \to \mathbb{R}$ that satisfy $f(a) \neq f(b)$. If a continuous function $f : I \to \mathbb{R}$ satisfies $f(a) = f(b)$ instead, then the Intermediate Value Theorem does not apply for this f, and we would have to employ a different argument without using the theorem.*

2. Let f be a continuous function defined on a closed, bounded interval $I = [a, b]$. Assume that f is one-to-one. Let m (respectively, M) be the minimum (respectively, maximum) of f. Then by Exercise 1, we know that either $f(a) = m$ and $f(b) = M$, or $f(a) = M$ and $f(b) = m$. If $f(a) = m$ and $f(b) = M$, then show that f is strictly monotone increasing. If $f(a) = M$ and $f(a) = m$, then show that f is strictly monotone decreasing.

   Hint: You need to use the Intermediate Value Theorem and argue by contradiction.

   Remark (from the professor)**.** *Exercise 2 basically states that the only way for a continuous function to be one-to-one is to be strictly monotone. In other words, only a strictly monotone continuous function has its inverse. This is actually a theorem in our textbook.*

   Proof. First, we will prove that, if $f(a) = m$ and $f(b) = M$, then f is strictly monotone increasing. Suppose instead that f is not strictly monotone increasing on $I = [a, b]$. Then there exists $x_1, x_2 \in I$ such that $x_1 < x_2$ and $f(x_1) \geq f(x_2)$. Now, $f(x_1) \geq f(x_2)$ splits into the two cases $f(x_1) = f(x_2)$ and $f(x_1) > f(x_2)$, and we seek to obtain a contradiction in each of these cases.

   - Case 1: Suppose $x_1 < x_2$ and $f(x_1) = f(x_2)$. Note that $x_1 < x_2$ implies $x_1 \neq x_2$. But we also have $f(x_1) = f(x_2)$ by our assumption for this case. This contradicts our assumption that f is one-to-one.

- Case 2: Suppose $x_1 < x_2$ and $f(x_1) > f(x_2)$. First, we note that $x_1, x_2 \in I$ with $x_1 < x_2$ is equivalent to saying $a \leq x_1 < x_2 \leq b$. In fact, we claim that x_1, x_2 cannot lie on the endpoints of the interval I; in other words, all the inequalities are strict: $a < x_1 < x_2 < b$. To prove our claim, we will argue that the cases $x_1 = a$ and $x_2 = b$ are impossible. Indeed, if $x_1 = a$, then we would have

$$\begin{aligned} f(x_2) &< f(x_1) \\ &= f(a) \\ &= m, \end{aligned}$$

which contradicts the assumption that m is the minimum of f on I. If $x_2 = b$, then we would have

$$\begin{aligned} f(x_1) &> f(x_2) \\ &= f(b) \\ &= M, \end{aligned}$$

which contradicts the assumption that M is the maximum of f on I. So we have justified our claim. With all this said, the rest of this argument will be similar to the proof of Exercise 1. Note that our chain of inequalities

$$\begin{aligned} f(a) = m &\leq f(x_2) \\ &< f(x_1) \\ &\leq M = f(b) \end{aligned}$$

implies

$$\begin{aligned} f(a) &\neq f(x_1), \\ f(x_1) &\neq f(x_2), \\ f(x_2) &\neq f(b), \end{aligned}$$

which allows us to apply the Intermediate Value Theorem on the closed intervals $[a, x_1], [x_1, x_2], [x_2, b]$, although we only require any two of these three closed intervals to complete our argument for this case. For instance, I choose to work with $[a, x_1]$ and $[x_1, x_2]$. Let γ be some number such that $f(x_2) = \max\{f(a), f(x_2)\} < \gamma < f(x_1)$. By the Intermediate Value Theorem, there exists $c_1 \in [a, x_1]$ such that $f(c_1) = \gamma$. Also by the Intermediate Value Theorem, there exists $c_2 \in [x_1, x_2]$ such that $f(c_2) = \gamma$. Note that $c_1 < x_1 < c_2$ implies $c_1 \neq c_2$. But we also have $f(c_1) = \gamma = f(c_2)$. This contradicts our assumption that f is one-to-one.

From these two cases, we have established that $x_1 < x_2$ and $f(x_1) \geq f(x_2)$ is not possible, given the hypotheses on f. So we must conclude that, if $x_1 < x_2$, then $f(x_1) < f(x_2)$. In other words, we conclude that f is strictly monotone increasing.

Next, we will prove that, if $f(a) = M$ and $f(b) = m$, then f is strictly monotone decreasing. Suppose instead that f is not strictly monotone decreasing on $I = [a, b]$. Then there exists $x_1, x_2 \in I$ such that $x_1 < x_2$ and $f(x_1) \leq f(x_2)$. Now, $f(x_1) \leq f(x_2)$ splits into the two cases $f(x_1) = f(x_2)$ and $f(x_1) < f(x_2)$, and we seek to obtain a contradiction in each of these cases.

- Case 1: Suppose $x_1 < x_2$ and $f(x_1) = f(x_2)$. Note that $x_1 < x_2$ implies $x_1 \neq x_2$. But we also have $f(x_1) = f(x_2)$ by our assumption for this case. This contradicts our assumption that f is one-to-one.
- Case 2: Suppose $x_1 < x_2$ and $f(x_1) < f(x_2)$. First, we note that $x_1, x_2 \in I$ with $x_1 < x_2$ is equivalent to saying $a \leq x_1 < x_2 \leq b$. In fact, we claim that x_1, x_2 cannot lie on the endpoints of the interval I; in other words, all the inequalities are strict: $a < x_1 < x_2 < b$. To prove our claim, we will argue that the cases $x_1 = a$ and $x_2 = b$ are impossible. Indeed, if $x_1 = a$, then we would have

$$\begin{aligned} f(x_2) &> f(x_1) \\ &= f(a) \\ &= M, \end{aligned}$$

which contradicts the assumption that M is the maximum of f on I. If $x_2 = b$, then we would have

$$\begin{aligned} f(x_1) &< f(x_2) \\ &= f(b) \\ &= m, \end{aligned}$$

which contradicts the assumption that m is the minimum of f on I. So we have justified our claim. With all this said, the rest of this argument will be similar to the proof of Exercise 1. Note that our chain of inequalities

$$\begin{aligned} f(a) = M &\geq f(x_2) \\ &> f(x_1) \\ &\geq m = f(b) \end{aligned}$$

implies

$$f(a) \neq f(x_1),$$
$$f(x_1) \neq f(x_2),$$
$$f(x_2) \neq f(b),$$

which allows us to apply the Intermediate Value Theorem on the closed intervals $[a, x_1], [x_1, x_2], [x_2, b]$, although we only require any two of these three closed intervals to complete our argument for this case. For instance, I choose to work with $[a, x_1]$ and $[x_1, x_2]$. Let γ be some number such that $f(x_2) = \min\{f(a), f(x_2)\} > \gamma > f(x_1)$. By the Intermediate Value Theorem, there exists $c_1 \in [a, x_1]$ such that $f(c_1) = \gamma$. Also by the Intermediate Value Theorem, there exists $c_2 \in [x_1, x_2]$ such that $f(c_2) = \gamma$. Note that $c_1 < x_1 < c_2$ implies $c_1 \neq c_2$. But we also have $f(c_1) = \gamma = f(c_2)$. This contradicts our assumption that f is one-to-one.

From these two cases, we have established that $x_1 < x_2$ and $f(x_1) \geq f(x_2)$ is not possible, given the hypotheses on f. So we must conclude that, if $x_1 < x_2$, then $f(x_1) > f(x_2)$. In other words, we conclude that f is strictly monotone decreasing. □

Remark. *It may be a good idea to draw a picture of some graph of f that meets all the stated criteria, as it may help you formulate a correct proof. For instance, when I was writing this proof, I had to draw some graph of f that satisfies $f(a) = m$, $f(b) = M$, $a < x_1 < x_2 < b$, and $f(x_1) > f(x_2)$; staring at this graph would prompt me to employ the Intermediate Value Theorem like one has to for Exercise 1.*

3. Consider $f : I \to \mathbb{R}$. Let J be an interval containing $f(I)$. Let $g : J \to \mathbb{R}$. Consider the composition of f and g, i.e. $g \circ f : I \to \mathbb{R}$. Show that:

   (1) If both f and g are strictly monotone increasing, then $g \circ f : I \to \mathbb{R}$ is strictly monotone increasing.

   Proof. Let $x_1, x_2 \in I$ be such that $x_1 < x_2$. Note that $x_1, x_2 \in I$ implies $f(x_1), f(x_2) \in f(I) \subset J$. Since f is strictly monotone increasing, we have $f(x_1) < f(x_2)$. Since g is strictly monotone increasing, we have $g \circ f(x_1) < g \circ f(x_2)$. Therefore, $g \circ f$ is strictly monotone increasing. □

   (2) If both f and g are strictly monotone decreasing, then $g \circ f : I \to \mathbb{R}$ is strictly monotone increasing.

   Proof. Let $x_1, x_2 \in I$ be such that $x_1 < x_2$. Note that $x_1, x_2 \in I$ implies $f(x_1), f(x_2) \in f(I) \subset J$. Since $f : I \to \mathbb{R}$ is strictly monotone increasing, we have $f(x_1) > f(x_2)$, which is equivalent to $f(x_2) < f(x_1)$. Since $g : J \to \mathbb{R}$ is strictly monotone decreasing, we have $g \circ f(x_2) > g \circ f(x_1)$, which is equivalent to $g \circ f(x_1) < g \circ f(x_2)$. Therefore, $g \circ f : I \to \mathbb{R}$ is strictly monotone increasing. □

   (3) If one of f and g is strictly monotone increasing and the other is strictly monotone decreasing, then $g \circ f : I \to \mathbb{R}$ is strictly monotone decreasing.

   Proof. Let $x_1, x_2 \in I$ be such that $x_1 < x_2$. Note that $x_1, x_2 \in I$ implies $f(x_1), f(x_2) \in f(I) \subset J$. To say that one of f and g is strictly monotone increasing and the other is strictly monotone decreasing is equivalent to saying f is strictly monotone increasing (respectively, decreasing) and g is strictly monotone decreasing (respectively, increasing). This motivates us to prove by cases.

   - Case 1: Suppose $f : I \to \mathbb{R}$ is strictly monotone increasing and $g : I \to \mathbb{R}$ is strictly monotone decreasing. Since $f : I \to \mathbb{R}$ is strictly monotone increasing, we have $f(x_1) < f(x_2)$. Since $g : J \to \mathbb{R}$ is strictly monotone decreasing, we have $g \circ f(x_1) > g \circ f(x_2)$.
   - Case 2: Suppose $f : I \to \mathbb{R}$ is strictly monotone decreasing and $g : I \to \mathbb{R}$ is strictly monotone increasing. Since $f : I \to \mathbb{R}$ is strictly monotone decreasing, we have $f(x_1) > f(x_2)$, which is equivalent to $f(x_2) < f(x_1)$. Since $g : J \to \mathbb{R}$ is strictly monotone increasing, we have $g \circ f(x_2) < g \circ f(x_1)$, which is equivalent to $g \circ f(x_1) > g \circ f(x_2)$.

   In both cases, we obtained $g \circ f(x_1) > g \circ f(x_2)$. Therefore, $g \circ f : I \to \mathbb{R}$ is strictly monotone decreasing. □

4. Let $f : \mathbb{R} \to \mathbb{R}$ to be a continuous, strictly monotone function. Let $f(\mathbb{R}) = \mathbb{R}$. Then its inverse f^{-1} is defined on $\mathbb{R}$. Show that $f^{-1} : \mathbb{R} \to \mathbb{R}$ continuous on $\mathbb{R}$.

Hint: Note that Theorem 2.10 only deals with the case where the domain is a closed, bounded interval. So you cannot apply it directly to this exercise. But you may follow the proof of continuity for $g : [0, \infty) \to [0, \infty)$ defined by $g(x) = \sqrt{x}$, which is the inverse of $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^2$.

Proof. We will show that f^{-1} is continuous on $\mathbb{R}$. Let $\gamma \in \mathbb{R}$ be given; that is, we will construct an argument that holds for all $\gamma \in \mathbb{R}$. Restrict f^{-1} to the closed interval $[-M, M] \subset \mathbb{R}$ for some $M > \gamma$ (so that we have $\gamma \in [-M, M]$), and call this new function $f^{-1}|_{[-M,M]}$. Since f is strictly monotone, the interval $[f^{-1}(-M), f^{-1}(M)]$ makes sense, and $f^{-1}|_{[-M,M]}$ is the inverse of $f|_{[f^{-1}(-M), f^{-1}(M)]}$. Since we were given that f is a continuous, strictly monotone function on $\mathbb{R}$, it follows in particular that f is a continuous, strictly monotone function on $[f^{-1}(-M), f^{-1}(M)]$. By Theorem 2.10 (page 101) of the professor's lecture notes, its inverse f^{-1} is also a continuous, strictly monotone function on $[-M, M]$. In particular, f^{-1} is a continuous, strictly monotone function at some $\gamma \in [-M, M]$. But we actually recall $\gamma \in \mathbb{R}$ from earlier, not just $\gamma \in [-M, M]$. Since $\gamma \in \mathbb{R}$ is arbitrary (in other words, we argued for all $\gamma \in \mathbb{R}$), we conclude that f^{-1} is a continuous, strictly monotone function on $\mathbb{R}$. □

Remark. *For ease of notation, you can let* $g = f^{-1}$ *and use* g *for the rest of the proof, and the professor certainly does this in his lecture notes. For me, as I stuck with* f^{-1} *all the way, I chose not to introduce* g *in my solutions, because I do not believe in introducing more notation unless absolutely necessary. In any case, feel free to use whatever notation that will work best for you, but stay consistent with your notation!*