HOMEWORK ASSIGNMENT NINE

MATH 150A, WINTER 2020

1. We've proved in class the follow lemma: *between any two real numbers there is a rational number*. Use this fact to show by definition that $\mathbb{Q}' = \mathbb{R}$. In other words, every real number is a limit point of the set of rational numbers.

2. Use the following strategy to show continuity of $f(x)$. Note for $a > 1$, we've showed that $f(x) = a^x : \mathbb{R} \to \mathbb{R}$ is strictly monotone increasing on $\mathbb{R}$. Show that:

(1) the sequence $\{a^{\frac{1}{n}}\}$ is monotone decreasing and bounded below by 1.

(2) 1 is the greatest lower bound of $\{a^{\frac{1}{n}}\}$. Thus,

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1.$$

Then show that for any $a > 0$, it holds that $\lim_{n\to\infty} a^{\frac{1}{n}} = \lim_{n\to\infty} a^{-\frac{1}{n}} = 1$.

(3) Use part (2) above and the monotonicity of $f(x) = a^x$ to show by definition that $f(x) = a^x$ is continuous at $x = 0$.

(4) Use the formula $a^x a^y = a^{x+y}$ for all $x, y \in \mathbb{R}$ and part (3) above to show that $f(x)$ is continuous on $\mathbb{R}$.

3. We've showed in class that or all $a > 0$ and all $x, y \in \mathbb{R}$ that

$$a^{x+y} = a^x a^y \text{ and } (a^x)^y = a^{xy}.$$

Fix $a > 0$ and $a \neq 1$. Then we can define $f(x) = \log_a(x) : \mathbb{R}_+ \to \mathbb{R}$ to be the inverse function of $a^x : \mathbb{R} \to \mathbb{R}_+$. Use formulas above and defintion of $f(x)$ to show that

$$\log_a(xy) = \log_a(x) + \log_a(y) \text{ and } \log_a(x^y) = y \log_a(x) \ \forall x, y \in \mathbb{R}_+.$$

4. For each $a \in \mathbb{R}$ and $x > 0$, we have defined what is x^a. Fix $a \in \mathbb{R}$ and we consider x^a as a function in $x \in \mathbb{R}_+$, i.e.

$$f_a(x) = x^a : \mathbb{R}_+ \to \mathbb{R}.$$

(1) Use problem 3 to show that $x^a = e^{a \ln(x)}$ for all $x > 0$.

(2) Use the above equality to show that $f_a(x) = x^a$ is continuous on $\mathbb{R}_+$.

(3) Again use the equality from part (1) to show that: if $a > 0$, then $f_a(x) = a^x$ is strictly monotone increasing; if $a < 0$, then $f_a(x) = x^a$ is strictly monotone decreasing.

(4) By part (3), we know that $f_a(x)$ has an inverse for any $a \neq 0$. Compute its inverse.