MATH 150A discussion
University of California, Riverside

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Homework 9 solutions

Reminder: If you choose to view these solutions, please only do so whenever you cannot make further progress on solving a problem. Make sure you understand firmly what is being written, and make sure that you are writing your own proof in your own words!

1. We proved in class the following lemma: "Between any two real numbers there is a rational number." Use this fact to show by definition that $\mathbb{Q}' = \mathbb{R}$. In other words, every real number is a limit point of the set of rational numbers.

   Proof. Let $\epsilon > 0$ be givenn, and let $x \in \mathbb{R}$ be arbitrary. Then we also have $x - \epsilon, x + \epsilon \in \mathbb{R}$. By the fact given in the problem statement, there exists $y \in (x - \epsilon, x) \cap \mathbb{Q}$ or $y \in (x, x + \epsilon) \cap \mathbb{Q}$. In other words, there exists $y \in \mathbb{Q} \setminus \{x\}$ such that $|y - x| < \epsilon$. Therefore, $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$. Finally, as $x \in \mathbb{R}$ is chosen arbitrarily, we conclude $\mathbb{Q}' = \mathbb{R}$. □

2. Use the following procedure to show that f is continuous. Note for $a > 1$, we showed that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a^x$ is strictly monotone increasing on $\mathbb{R}$.

   (1) Show that the sequence $\{a^{\frac{1}{n}}\}_{n=1}^{\infty}$ is monotone decreasing and bounded below by 1.

   Proof. First, we will show that $\{a^{\frac{1}{n}}\}_{n=1}^{\infty}$ is monotone decreasing. For all $n \in \mathbb{Z}_+$, we have $n < n + 1$, which is equivalent to $\frac{1}{n+1} < \frac{1}{n}$. According to Lemma 2 of Section 2.6 (page 109) of the professor's lecture notes, f is strictly monotone increasing on $\mathbb{R}$. So we have $f(\frac{1}{n+1}) < f(\frac{1}{n})$. In other words, we have

   $$\begin{aligned} a^{\frac{1}{n}} &= f\left(\frac{1}{n}\right) \\ &> f\left(\frac{1}{n+1}\right) \\ &= a^{\frac{1}{n+1}} \end{aligned}$$

   for all $n \in \mathbb{Z}_+$. Therefore, $\{a^{\frac{1}{n}}\}_{n=1}^{\infty}$ is monotone decreasing.

   Next, we will show that $\{a^{\frac{1}{n}}\}_{n=1}^{\infty}$ is bounded below by 1. That is, we will show $a^{\frac{1}{n}} > 1$ for all $n \in \mathbb{Z}_+$. Suppose instead there exists $n \in \mathbb{Z}_+$ such that $a^{\frac{1}{n}} \leq 1$. Recall that power functions are increasing functions for all $n \in \mathbb{Z}_+$; that is, if $x, y \in \mathbb{R}$ satisfies $x < y$, then we also have $x^n < y^n$ for all $n \in \mathbb{Z}_+$. With all this said, we have

   $$\begin{aligned} a &= (a^{\frac{1}{n}})^n \\ &\leq 1^n \\ &= 1, \end{aligned}$$

   which contradicts our assumption $a > 1$. Therefore, we conclude $a^{\frac{1}{n}} > 1$ for all $n \in \mathbb{Z}_+$; that is, $\{a^{\frac{1}{n}}\}_{n=1}^{\infty}$ is bounded below by 1. □

   Remark. *The proof showing that $\{a^{\frac{1}{n}}\}_{n=1}^{\infty}$ is bounded below by 1 is taken directly from the proof of Lemma 2 of Section 2.6 (page 109) of the professor's lecture notes, although I paraphrased it in my own words.*

   (2) Show that 1 is the greatest lower bound of $\{a^{\frac{1}{n}}\}_{n=1}^{\infty}$. Thus,

   $$\lim_{n\to\infty} a^{\frac{1}{n}} = 1.$$

   Then show that for any $a > 0$, it holds that $\lim_{n\to\infty} a^{\frac{1}{n}} = \lim_{n\to\infty} a^{-\frac{1}{n}} = 1$.

   Proof. We showed already in part (1) that 1 is a lower bound of $\{a^{\frac{1}{n}}\}_{n=1}^{\infty}$. To show that 1 is the greatest lower bound of $\{a^{\frac{1}{n}}\}_{n=1}^{\infty}$, it remains to show that, if b is any lower bound of $\{a^{\frac{1}{n}}\}_{n=1}^{\infty}$, then $b \leq 1$. Suppose instead we have $b > 1$. As b is a lower bound of $\{a^{\frac{1}{n}}\}_{n=1}^{\infty}$, we have $a^{\frac{1}{n}} > b$, or equivalently $a > b^n$, for all $n \in \mathbb{Z}_+$. Since $b > 1$, the sequence $\{b^n\}_{n=1}^{\infty}$ is unbounded, and so it is divergent, which implies that the statement $a > b^n$ for all $n \in \mathbb{Z}_+$ is actually a contradiction. Therefore, we must have $b \leq 1$, meaning that 1 is the greatest lower bound of $\{a^{\frac{1}{n}}\}_{n=1}^{\infty}$. □

   Remark. *I am honestly not sure how to show $\lim_{n\to\infty} a^{\frac{1}{n}} = \lim_{n\to\infty} a^{-\frac{1}{n}} = 1$. Please ask the professor in his online office hours.*

   (3) Use part (2) above and the monotonicity of f to show by definition that f is continuous at $x = 0$.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ defined by $x_n = \frac{1}{n}$. Then we have

$$\begin{aligned}\lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{1}{n}\\ &= 0.\end{aligned}$$

If $a > 0$, then, by part (2), we also have

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} f\left(\frac{1}{n}\right)\\ &= \lim_{n\to\infty} a^{\frac{1}{n}}\\ &= 1\\ &= a^0\\ &= f(0).\end{aligned}$$

Similarly, let $\{y_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ defined by $y_n = -\frac{1}{n}$. Then we have

$$\begin{aligned}\lim_{n\to\infty} y_n &= \lim_{n\to\infty} \left(-\frac{1}{n}\right)\\ &= -\lim_{n\to\infty} \frac{1}{n}\\ &= 0.\end{aligned}$$

If $a > 0$, then, by part (2), we also have

$$\begin{aligned}\lim_{n\to\infty} f(y_n) &= \lim_{n\to\infty} f\left(-\frac{1}{n}\right)\\ &= \lim_{n\to\infty} a^{-\frac{1}{n}}\\ &= 1\\ &= a^0\\ &= f(0).\end{aligned}$$

Therefore, f is continuous at $x = 0$, according to Definition 2.2 (page 62) of the professor's lecture notes. □

(4) Use the formula $a^x a^y = a^{x+y}$ for all $x, y \in \mathbb{R}$ and part (3) above to show that f is continuous on $\mathbb{R}$.

Proof. To show that f is continuous on $\mathbb{R}$, we need to show that f is continuous at any $\xi \in \mathbb{R}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence that satisfies $\lim_{n\to\infty} x_n = \xi$. If we let $y_n := x_n - \xi$, then the Sum Law for limits implies

$$\begin{aligned}\lim_{n\to\infty} y_n &= \lim_{n\to\infty} (x_n - \xi)\\ &= \lim_{n\to\infty} x_n + \lim_{n\to\infty} (-\xi)\\ &= \xi + (-\xi)\\ &= 0.\end{aligned}$$

As we know already from part (3) that f is continuous at 0, we also have $\lim_{n\to\infty} f(y_n) = f(0)$. Now, applying the formula $a^x a^y = a^{x+y}$, we have

$$\begin{aligned}f(x_n) &= a^{x_n}\\ &= a^{\xi + (x_n - \xi)}\\ &= a^{\xi + y_n}\\ &= a^{\xi} a^{y_n}\\ &= f(\xi) f(y_n),\end{aligned}$$

and so by the Product Law for limits we have

$$\begin{aligned}\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} (f(\xi) f(y_n))\\ &= (\lim_{n\to\infty} f(\xi))(\lim_{n\to\infty} f(y_n))\\ &= f(\xi) \lim_{n\to\infty} f(y_n)\\ &= f(\xi) f(0)\\ &= f(\xi) a^0\\ &= f(\xi) \cdot 1\\ &= f(\xi).\end{aligned}$$

Therefore, f is continuous at $\xi \in \mathbb{R}$, according to Definition 2.2 (page 62) of the professor's lecture notes. So f is continuous on $\mathbb{R}$. □

Remark. *For the professor's version, which is simpler than mine, see Theorem 2.12 (page 116) of the professor's lecture notes.*

Alternate proof (unnecessary, difficult, perhaps confusing). This is an ϵ-δ argument. To show that f is continuous on $\mathbb{R}$, we need to show that f is continuous at any $\xi \in \mathbb{R}$. Let $\epsilon > 0$ be given, and choose $\delta = \log_a(\frac{\epsilon}{a^\xi} + 1)$. (Note that $f(\xi) = a^\xi$ is non-zero because exponential functions are always positive.) Then, for all $x \in (\xi - \delta, \xi + \delta)$ (or if $|x - \xi| < \delta$), we can apply the formula $a^x a^y = a^{x+y}$ to obtain

$$\begin{aligned} f(x) - f(\xi) &= a^x - a^\xi \\ &= a^{\xi+(x-\xi)} - a^\xi \\ &= a^\xi a^{x-\xi} - a^\xi \\ &= a^\xi(a^{x-\xi} - 1). \end{aligned}$$

Next, we claim that, if $a > 1$, then the inequality

$$|a^y - 1| \leq a^{|y|} - 1$$

holds for all $y \in \mathbb{R}$. To prove this claim, we should separate our argument into three distinct cases: $y > 0$, $y = 0$, $y < 0$.

- Case 1: Suppose $y > 0$. Then we have $|y| = y$. Since f is strictly monotone increasing, we have $f(y) > f(0)$, or equivalently

  $$\begin{aligned} a^y &= f(y) \\ &> f(0) \\ &= a^0 \\ &= 1, \end{aligned}$$

  which is in turn equivalent to $a^y - 1 > 0$, which implies $|a^y - 1| = a^y - 1$. So we conclude

  $$\begin{aligned} |a^y - 1| &= a^y - 1 \\ &= a^{|y|} - 1, \end{aligned}$$

  which is the equality case of the inequality.
- Case 2: Suppose $y = 0$. Then we have

  $$\begin{aligned} a^y - 1 &= a^0 - 1 \\ &= 1 - 1 \\ &= 0, \end{aligned}$$

  and so we conclude trivially

  $$\begin{aligned} |a^y - 1| &= |0| \\ &= 0 \\ &= a^y - 1, \end{aligned}$$

  which is the equality case of the inequality.
- Case 3: Suppose $y < 0$. Then we have $|y| = -y$. Since f is strictly monotone increasing, we have $f(y) < f(0)$, or equivalently

  $$\begin{aligned} a^y &= f(y) \\ &< f(0) \\ &= a^0 \\ &= 1, \end{aligned}$$

  which is in turn equivalent to $a^y - 1 < 0$, which implies $|a^y - 1| = -(a^y - 1)$. So we conclude

  $$\begin{aligned} |a^y - 1| &= -(a^y - 1) \\ &= 1 - a^y \\ &\leq a^{-y} - 1 \\ &= a^{|y|} - 1, \end{aligned}$$

which is the inequality case of the inequality. But to establish our desired claim for this case, we have just used

$$1 - a^y \leq a^{-y} - 1,$$

which is algebraically equivalent to

$$\frac{a^y + a^{-y}}{2} \geq 1,$$

but this inequality is not obvious (except maybe visually, when you view the graph of $\frac{a^y+a^{-y}}{2}$). So we need to justify this inequality. Let $p, q \in \mathbb{R}_+$ be arbitrary. Then we have

$$\begin{aligned} 0 &\leq (p-q)^2 \\ &= p^2 - 2pq + q^2 \\ &= (p^2 + 2pq + q^2) - 4pq \\ &= (p+q)^2 - 4pq, \end{aligned}$$

which we can algebraically rearrange to deduce

$$\frac{(p+q)^2}{4} \geq pq.$$

We take the square root of both sides (and the inequality sign does not flip because the square root function is a strictly monotone increasing function on its domain $[0, \infty)$) to further obtain

$$\frac{p+q}{2} \geq \sqrt{pq}.$$

(This is a special inequality called the *inequality of arithmetic and geometric means*.) Finally, as we made $p, q \in \mathbb{R}_+$ to be arbitrary, we can let $p := a^y$ and $q := a^{-y}$. Then our inequality becomes

$$\begin{aligned} \frac{a^y + a^{-y}}{2} &\geq \sqrt{a^y a^{-y}} \\ &= \sqrt{a^{y+(-y)}} \\ &= \sqrt{a^0} \\ &= \sqrt{1} \\ &= 1, \end{aligned}$$

and this proves the inequality that we originally said was not obvious.

In all the cases, we have proved our claim. For $y := x - \xi$, our claim becomes

$$|a^{x-\xi} - 1| \leq a^{|x-\xi|} - 1.$$

Also, as f is strictly monotone increasing on $\mathbb{R}$ and $|x - \xi| < \delta$, we have $f(|x - \xi|) < f(\delta)$, or equivalently

$$\begin{aligned} a^{|x-\xi|} &= f(|x-\xi|) \\ &< f(\delta) \\ &= a^{\delta}. \end{aligned}$$

Finally, we have

$$\begin{aligned} |f(x) - f(\xi)| &= |a^{\xi}(a^{x-\xi} - 1)| \\ &= a^{\xi}|a^{x-\xi} - 1| \\ &\leq a^{\xi}(a^{|x-\xi|} - 1) \\ &< a^{\xi}(a^{\delta} - 1) \\ &= a^{\xi}(a^{\log_a(\frac{\epsilon}{a^{\xi}}+1)} - 1) \\ &= a^{\xi}\left(\left(\frac{\epsilon}{a^{\xi}} + 1\right) - 1\right) \\ &= \epsilon. \end{aligned}$$

Therefore, f is continuous at $\xi \in \mathbb{R}$, and so f is continuous on $\mathbb{R}$. □

Remark. *The alternate proof using the ϵ-δ definition of continuity for the exponential function is presented here only for your amusement. If you are studying for the final exam, please ignore this proof. I guarantee that you will not see this proof on your final exam.*

3. We showed in class that

$$a^{x+y} = a^x a^y,$$
$$(a^x)^y = a^{xy}.$$

for all $a > 0$ and all $x, y \in \mathbb{R}$. Fix $a > 0$ and $a \neq 1$. Then $g : \mathbb{R}_+ \to \mathbb{R}$ defined by $g(x) = \log_a(x)$ is the inverse function of $f : \mathbb{R} \to \mathbb{R}_+$ given by $f(x) = a^x$. Use the formulas above and the definition of f to show

$$\log_a(xy) = \log_a(x) + \log_a(y),$$
$$\log_a(x^y) = y \log_a(x)$$

for all $x, y \in \mathbb{R}_+$.

Proof. Define $f : \mathbb{R} \to \mathbb{R}_+$ and $g : \mathbb{R}_+ \to \mathbb{R}$ by

$$f(x) := a^x,$$
$$g(x) := \log_a(x).$$

Then f and g are inverses of each other, which means we have

$$\begin{aligned} a^{\log_a(x)} &= a^{g(x)} \\ &= f \circ g(x) \\ &= x \end{aligned}$$

for all $x \in \mathbb{R}_+$ and

$$\begin{aligned} \log_a(a^x) &= \log_a(f(x)) \\ &= g \circ f(x) \\ &= x \end{aligned}$$

for all $x \in \mathbb{R}$. So, for all $x, y \in \mathbb{R}_+$, we have

$$\begin{aligned} \log_a(xy) &= \log_a(a^{\log_a x} a^{\log_a y}) \\ &= \log_a(a^{\log_a(x)+\log_a(y)}) \\ &= \log_a(x) + \log_a(y), \end{aligned}$$

as desired. Also, for all $x \in \mathbb{R}_+$ and $y \in \mathbb{R}$, we have

$$\begin{aligned} \log_a(x^y) &= \log_a((a^{\log_a(x)})^y) \\ &= \log_a(a^{y \log_a(x)}) \\ &= y \log_a(x), \end{aligned}$$

as desired. □

4. Fix $a \in \mathbb{R}$ and define $f_a : \mathbb{R}_+ \to \mathbb{R}$ by $f_a(x) = x^a$.

   (1) Use Exercise 3 to show $x^a = e^{a \ln(x)}$ for all $x > 0$.

   Proof. Note that we have
   $$e^{\ln(x)} = x$$
   for all $x \in \mathbb{R}$. By Exercise 3, we have
   $$\ln(x^a) = a \ln(x)$$
   for all $x \in \mathbb{R}$ and for all $a \in \mathbb{R}$. Therefore, we have
   $$\begin{aligned} e^{a \ln(x)} &= e^{\ln(x^a)} \\ &= x^a, \end{aligned}$$
   as desired. □

   Remark. *Recall that the natural logarithm is defined as the logarithm with base e; that is,* $\ln y = \log_e y$ *for all* $y \in \mathbb{R}_+$.

   (2) Use the equality from part (1) show that f_a is continuous on $\mathbb{R}_+$.

Proof. Define $g_a : \mathbb{R} \to \mathbb{R}$ by $g_a(x) = ax$, $h_1 : \mathbb{R}_+ \to \mathbb{R}$, and $h_2 : \mathbb{R} \to \mathbb{R}_+$ by

$$\begin{aligned} g_a(x) &= ax, \\ h_1(x) &= \ln(x), \\ h_2(x) &= e^x. \end{aligned}$$

It is already shown in both your lecture and Exercise 3 that exponential functions are continuous, which means h_2 is continuous. As h_2 is continuous on $\mathbb{R}$, it is in particular continuous on a bounded domain $I \subset \mathbb{R}$. Since h_2 are also strictly monotone increasing on $\mathbb{R}$, by Theorem 2.10 (page 101) of the professor's lecture notes, the inverse h_1 is continuous on $f^{-1}(I) \subset \mathbb{R}_+$. Then we can extend this argument to show that h_1 is continuous on all of $\mathbb{R}_+$. Finally, for any fixed $a \in \mathbb{R}$, it was shown already in your lecture that power functions are continuous; that is, g_a is continuous. Now, by part (1), we have

$$\begin{aligned} f_a(x) &= x^a \\ &= e^{a \ln x} \\ &= e^{ah_1(x)} \\ &= e^{g_a \circ h_1(x)} \\ &= h_2 \circ g_a \circ h_1(x) \end{aligned}$$

for all $x \in \mathbb{R}_+$. So $f_a = h_2 \circ g_a \circ h_1$ is a composition of continuous functions. By Theorem 2.6 (pages 82-83) of the professor's lecture notes, the composition of continuous functions is a continuous function; in other words, f_a is continuous on $\mathbb{R}_+$. □

(3) Use the equality from part (1) again to show that:

(a) If $a > 0$, then f_a is strictly monotone increasing.

Proof. As in the proof of part (2), define $g_a : \mathbb{R} \to \mathbb{R}$ by $g_a(x) = ax$, $h_1 : \mathbb{R}_+ \to \mathbb{R}$, and $h_2 : \mathbb{R} \to \mathbb{R}_+$ by

$$\begin{aligned} g_a(x) &= ax, \\ h_1(x) &= \ln(x), \\ h_2(x) &= e^x. \end{aligned}$$

If $a > 0$, then g_a is a strictly monotone increasing function on $\mathbb{R}$. Also, h_1 is a strictly monotone increasing function on $\mathbb{R}_+$, and h_2 is a strictly monotone increasing function on $\mathbb{R}$. We also recall $f_a = h_2 \circ g_a \circ h_1$. By Exercise 3, part (1) of Homework 8, $g_a \circ h_1$ is strictly monotone decreasing. Furthermore, by that same homework exercise, we conclude that $f_a = h_2 \circ g_a \circ h_1$ is strictly monotone increasing. □

(b) If $a < 0$, then f_a is strictly monotone decreasing.

Proof. As in the proof of part (2), define $g_a : \mathbb{R} \to \mathbb{R}$ by $g_a(x) = ax$, $h_1 : \mathbb{R}_+ \to \mathbb{R}$, and $h_2 : \mathbb{R} \to \mathbb{R}_+$ by

$$\begin{aligned} g_a(x) &= ax, \\ h_1(x) &= \ln(x), \\ h_2(x) &= e^x. \end{aligned}$$

If $a < 0$, then g_a is a strictly monotone decreasing function on $\mathbb{R}$. Also, h_1 is a strictly monotone increasing function on $\mathbb{R}_+$, and h_2 is a strictly monotone increasing function on $\mathbb{R}$. We also recall $f_a = h_2 \circ g_a \circ h_1$. By Exercise 3, part (3) of Homework 8, $g_a \circ h_1$ is strictly monotone decreasing. Furthermore, by that same homework exercise, we conclude that $f_a = h_2 \circ g_a \circ h_1$ is strictly monotone decreasing. □

Remark. *I used the notations g_a, h_1, h_2 to write f_a as a composition of strictly monotone increasing/decreasing functions. My choice of notation is arbitrary; feel free to use your own notation for the functions if you would like.*

(4) By part (3), we know that f_a has an inverse for any $a \neq 0$. Compute its inverse.

Proof. Define $g_a : \mathbb{R} \to \mathbb{R}_+$ by $g_a(x) = x^{\frac{1}{a}}$. For all $x \in \mathbb{R}$, we have

$$\begin{aligned} f_a \circ g_a(x) &= f_a(x^{\frac{1}{a}}) \\ &= (x^{\frac{1}{a}})^a \\ &= x. \end{aligned}$$

For all $x \in \mathbb{R}_+$, we have

$$\begin{aligned} g_a \circ f_a(x) &= g_a(x^a) \\ &= (x^a)^{\frac{1}{a}} \\ &= x. \end{aligned}$$

So g_a is the inverse of f_a. □