MATH 150A - Intermediate Analysis

Chapter 1: Sequences & Limits

1.1 Introduction & Preparation

1.1.1. Introduction

What is analysis?

- It's a more advanced version of calculus.

What is calculus? In particular, what's the main object of study of calculus?

- Functions, specifically real variable & real valued functions, in other words (i.e.)

$$f: I \to \mathbb{R},$$

where $\mathbb{R}$ is the set of real numbers & $I \subseteq \mathbb{R}$ is an interval of real numbers, i.e.

$$I = [a,b],\ (a,b),\ (a,b],\ (-\infty, a),\ (b, +\infty),$$
$$\mathbb{R} = (-\infty, +\infty).$$

What properties of a function do we study in calculus?

- Continuity, differentiability, integrability

Similarly, in analysis, these are the main topics as well. Then, what's the difference between calculus such as MATH 9A/B/C and MATH 150A?

- Calculus mainly deals with computational stuff like all kinds of differentiation & integration techniques. However, calculus does not tell you why.

Analysis tells you why. Analysis care more about why can you do it, instead of how can you do it. In~~stead~~ other words, in analysis we treat everything rigorously in analysis.

Usually, if one wants to understand why, one needs to go back to the very beginning & understand the basic material at a much deeper level. So the first thing we need to do in ~~ab~~ analysis is to understand:

1) What is the set of real numbers, algebraically and geometrically?

In some sense, this is the most difficult question in analysis. We shall first give a brief & rough introduction. Then along the way, we shall have a better & better understanding. Then, we move on to understand:

2) What are sequences of real numbers?
In particular, what are convergent sequence?
How to compute the limit of a convergent sequence?

For example, we all know that
$\lim_{n\to\infty} \frac{1}{n} = 0$, $\lim_{n\to\infty} \frac{1}{2^n} = 0$, ...

But why? In fact, what do we mean by saying
$\lim_{n\to\infty} \frac{1}{n} = 0$?

By answering questions in 2), we will have a better understanding of question 1), i.e. we shall understand better ~~this~~ what is $\mathbb{R}$.

Here I want to emphasize that the notion of convergent sequence is the most difficult notion in college math! If you are able to understand this notion ~~thorug~~ thoroughly, then this course (even almost all college math courses) won't be a problem for you at all. We shall spend at least 2 or 3 lectures towards ~~understand~~ just understanding this notion. In fact, the

whole chapter one is devoted to this notion.
Though we do not study functions in chapter one, it's the corner stone of the whole course.
If you get lost at the very beginning, then it will only get more & more difficult for later stuff's. So you need to make sure to follow well at the beginning.

Once we finish sequences & limits, we shall move on to functions, in particular:

3) What are continuous functions?
What properties do they have?

To answer these questions, we rely heavily on our understanding of sequences & limits.
In the meantime, by answering these questions, we shall again have a better understanding of question 1) & questions in 2).

~~A~~ Concrete examples of questions in 3) are the following:

We all know $f: I \to \mathbb{R}$ is continuous at $x_0 \in I$ means $\lim_{x \to x_0} f(x) = f(x_0)$. Again, what does it really mean? How do we know that functions like $f(x) = x^n$, a^x, $\ln(x)$, $\sin(x)$, $e^{\sin(x)}$ ... are continuous?

Unfortunatelly, within the time of one-quarter, we are not able to discuss differentiability & Integrability of functions. But once you have a good ~~und~~ understanding of continuity, the discussion of differentiability & integrability follow in a similar pattern. Interested students may read the further materials themselves without much difficulties. Or you may take further courses like MATH 151 A/B/C.

However, in this class I hope that you all can do the following

- First, understand all the notions or definitions well
- Understand the properties or theorems deduced from those notions.
- Be able to apply them into solving concrete~~ly~~ problems.

To this end, I strongly recommend that you attend the class regually, go through all the written homework problems very carefully & make sure you can handle them well. Finally, read the book or the notes to digest the materials. If you have any questions, please come to my office hours.

This course ~~so~~ will give you ideas what is a real major math course? What is ~~me~~ math reasoning? Analysis is also important for applied math majors.

1.1.2 Preparation.

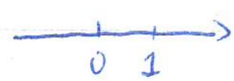
As we mentioned before, we first need to have a better understanding of the set of real numbers $\mathbb{R}$.

- Algebraically, $\mathbb{R}$ is a set of certain numbers such as ~~th~~ natural numbers ($\mathbb{N} = \{0, 1, 2, \dots\}$), integers ($\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$), rational numbers ($\mathbb{Q} = \{\frac{m}{n}, m, n \in \mathbb{Z}, n \neq 0\}$), & irrational numbers ~~so~~, i.e. $\sqrt{2}, e, \pi, \dots$

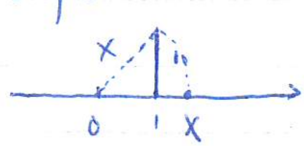
· Geometrically, $\mathbb{R}$ is a straight line with the origin, length of unit, and direction of positive numbers. We call such a line a real line

One of the most important facts in mathematics is:

> there is a 1-to-1 correspondence between the set of real numbers & the points on the real line. i.e. for each real number $x \in \mathbb{R}$, there corresponds a point on the real line; conversely, for each point on the real line, there corresponds a real number in $\mathbb{R}$.

This basic fact tell us that $\mathbb{R}$ & 

the real line are just different ways to describe the same math object. It's only true for $\mathbb{R}$, not $\mathbb{Q}$ or $\mathbb{Z}$.

For example, 

, use compasses & ruler we can find a point x on the real line.

i.e. : in other words.
s.t. : such that
e.g. : for example
$\mathbb{R}$: set of real numbers

$\mathbb{N}$: set of natural numbers
$\mathbb{Z}$: ... integers
$\mathbb{Q}$: ... rational number
$\mathbb{Z}_+$: ... positive integers

It turns out that there is no ~~no~~ number in the form $\frac{m}{n}$, $m,n \in \mathbb{Z}$ that corresponds to this point x. In fact, by Pythagrean Thm, we know $1^2+1^2=x^2 \Rightarrow x=\sqrt{2}$, which is a irrational number.

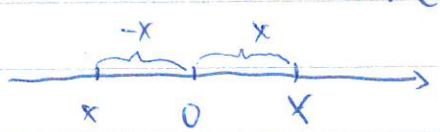
This fact also tells us one of the principles learning analysis, in fact all type of mathematics.

(I). One should always try to use geometrical pictures to visulize algebraic stuff. Conversely, we need to try to use algebraic stuff to accurately & rigorously describe geometric pictures.

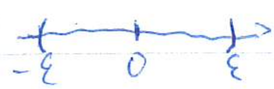
For example, we know that algebraically,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

is the absolute value of a real number x. Since x is also a point on the real line, there is a geometric way to interpret $|x|$: it's actually the distance between x & 0 on the real line

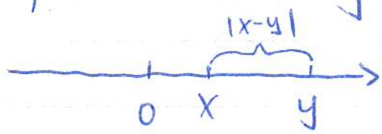


$-\varepsilon < x < \varepsilon \Leftrightarrow |x| < \varepsilon$



More generally, Let's consider $|x-y|$:

- algebraically, $|x-y|$ is the absolute value of the difference between two real numbers x & y.
- geometrically, $|x-y|$ is the distance between the two points x & y on the real line



Note that the algebraic relation $x<y$ means geometrically, the point x is to the left of the point y on the real line. Thus

$a<x<b$ means $x \in I=(a,b)$, means

Sometimes, we use $I=(a,b)=\{x: a<x<b\}$.

- algebraically, $(x_0-\varepsilon, x_0+\varepsilon)$ is an open interval, of all real numbers x s.t. $x_0-\varepsilon<x<x_0+\varepsilon$.

It's equivalent to $-\varepsilon<x-x_0<\varepsilon$, which is equivalent to $|x-x_0|<\varepsilon$. Thus

- geometrically, $(x_0-\varepsilon, x_0+\varepsilon)=\{x: x_0-\varepsilon<x<x_0+\varepsilon\}$ are all points whose distance with x_0 is less than ε.

We need another notion called bounded set of the real line.

<u>Definition 1.1</u>: Let $S \subseteq \mathbb{R}$ be a subset of the real number. We say that S is bounded if there are two real numbers $m \le M$ s.t.

$m \le x \le M$ for all x in S.

Remark: Geometrically, S is bounded meaning that all points in S are to the left of M & to the right of m.

$S \subseteq [m, M]$

We say that S is bounded below by m & m is a lower bound of S; S is bounded above by M & M is an upper bound of S.

If S is not bounded, then we say that S is unbounded.

Example 1:

① Finite set is bounded. S is finite if

$\Rightarrow$: implies | $\square$: end of a proof

it contains finite number of points. The number of points in a finite set is called its cardinality, denoted Card(S), which is a natural number.

Proof: S is finite $\Rightarrow$ Card(S) $= n \in \mathbb{N}$.

If $n=0$, then S is empty which is of course bounded since there is nothing in it.

If $n \geq 1$, then we may write S as

$$S = \{x_1, x_2, \cdots, x_n\}.$$

Then we set $m = \min\{x_1, x_2, \cdots, x_n\}$. Since m is ~~the~~ minimal one $\Rightarrow$ $x_i \geq m$ for all $i=1, \cdots, n$.

We set $M = \max\{x_1, \cdots, x_n\}$. Since M is the maximal one $\Rightarrow$ $x_i \leq M$ for all $i=1, \cdots, n$,

$\Rightarrow$ $m \leq x_i \leq M$, for all $i=1, \cdots, n$.

$\Rightarrow$ S is bounded with m being a lower-bound & M being a upper-bound. $\square$

② Finite intervals are bounded. Clearly a finite interval $I = (a,b), [a,b), (a,b], \text{ or } [a,b]$. In any case, $m=a$ & $M=b$ are a lower & upper bound of I, respectively $\Rightarrow$ I is bounded.

③. Example of unbounded set in $\mathbb{R}$

(12) But $|x+y|^2 = (x+y)^2 = x^2+y^2+2xy$

$(|x|+|y|)^2 = |x|^2+|y|^2 + 2|x|\cdot|y| = x^2+y^2+2|x\cdot y|$.

Cleanly, $xy \leq |xy| \Rightarrow 2xy \leq 2|xy| \Rightarrow |x+y|^2 \leq (|x|+|y|)^2$ □

Corollary: $||x|-|y|| \leq |x-y|$. Indeed, $|x| = |x-y+y| \leq |x-y| + |y|$

$\Rightarrow |x|-|y| \leq |x-y|$. Switching the role, we get $|y|-|x| \leq |x-y|$

$\Rightarrow ||x|-|y|| \leq |x-y|$ □

Claim: $\mathbb{Z}$ is unbounded.

- Idea of proof: here we shall a standard type of argument that will be used a lot throughout the quarter called argue by contradiction. Roughly speaking, we assume the conclusion that $\mathbb{Z}$ is bounded, from which we shall draw a contradiction. It then implies the assumption is false. Thus $\mathbb{Z}$ must be bounded.

- Proof: Assume that $\mathbb{Z}$ is bounded. By definition 1.1 $\mathbb{Z}$ is bounded above by some M, i.e.

$$n \leq M \text{ for all } n \text{ in } \mathbb{Z}.$$

But we know that for any M, there is a next greater integer $\lceil M \rceil$, in particular $\lceil M \rceil \in \mathbb{Z}$ & $\lceil M \rceil \geq M$.

$\Rightarrow \lceil M \rceil + 1 \in \mathbb{Z}$ & $\lceil M \rceil + 1 > M$ ①

But all points in $\mathbb{Z}$ are bounded above by M

Thus $\lceil M \rceil + 1 \leq M$ ②. ① & ② together are clearly a contradiction. Thus the assumption that $\mathbb{Z}$ is a bounded is false $\Rightarrow \mathbb{Z}$ is unbounded. □

Finally, triangle inequality

Theorem 1.1. For any x, y in $\mathbb{R}$, it holds that $|x+y| \leq |x|+|y|$

Proof: Since both $|x+y|$ & $|x|+|y|$ are non-negative, it's equivalent to show $|x+y|^2 \leq (|x|+|y|)^2$, cont'd at the top.

1.2 Limits of Sequences

1.2.1 Convergent Sequences

<u>Definition 1.2.</u> A sequence is a map from $\mathbb{Z}_+$ to $\mathbb{R}$, denoted

$$a: \mathbb{Z}_+ \longrightarrow \mathbb{R}$$
$$n \longmapsto a(n).$$

Here $\mathbb{Z}_+$ is the <u>domain</u> of a. The <u>range</u> of the sequence is the set of all distinct values $a(n)$. The <u>terms</u> of a sequence are the real numbers

$$a(1), a(2), \cdots, a(n), \cdots$$

which are usually denoted ~~as~~ with sub-scripts

$$a_1, a_2, \cdots, a_n, \cdots.$$

In particular, a_n is the nth term of the sequence. A sequence is usually denoted by $\{a_n\}_{n\geq 1}$, or simply $\{a_n\}$.

Example 1.

① $a: \mathbb{Z}_+ \to \mathbb{R}$, $a(n) \equiv c$ (meaning $a(n) = c$ for all n) is called a constant sequence. Although the sequence must be listed as $c, c, \cdots, c, \cdots$, the range of this sequence is a single point set $\{c\}$.

② $\{a_n = \frac{1}{n}\}_{n\geq 1}$ is a sequence with range $\{\frac{1}{n} : n \geq 1\}$.

③ $\{a_n = n\}_{n\geq 1}$ is a sequence with range $\{n : n\geq 1\} = \mathbb{Z}_+$.

Definition 1.3 We say a sequence $\{a_n\}$ converges to L as n goes to infinity if the following holds true:

for any $\varepsilon>0$, there exists an positive integer n_0 (i.e. $n_0 \in \mathbb{Z}_+$), s.t.

$$|a_n - L| < \varepsilon \quad \text{for all} \quad n \geq n_0$$

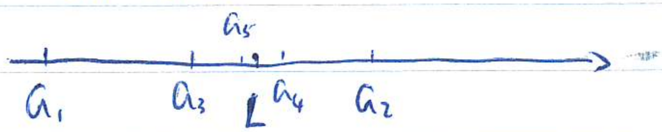
In this case, we say $\{a_n\}$ is convergent.

L is called the limit of $\{a_n\}$ & denoted $\lim_{n\to\infty} a_n = L$

Again, this is the most difficult notion of college math & we shall spend a lot of time understanding it. But first let's try to describe it geometrically.

Let's put everything on the real line.

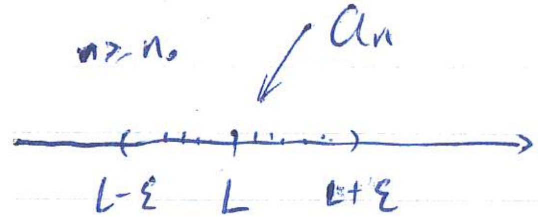
Roughly speaking, $\lim_{n\to\infty} a_n = L$ means that as $n\to\infty$, a_n tends to L



It describes the asymptotic behavior of a_n as $n\to\infty$.

But what do we really mean by saying a_n tends to L as $n\to\infty$? Precisely, it means that no matter how small $\varepsilon>0$ is, eventually, all a_n are within ε-distance to L, i.e. $|a_n - L| < \varepsilon$. What the rigorous meaning of "eventually all"? It means there is a $n_0 \geq 1$, that $|a_n - L| < \varepsilon$ for all

$n \geq n_0$.



Remark: ① From the definition, we may see that $\lim a_n = L$ concerns only the asympototic behavior of $\{a_n\}$. It doesn't concern the first finite number of terms. In other words, if

$$\lim_{n \to \infty} a_n = L$$

then the change of first finite number of terms won't affect this fact.

② The key thing in the definition 1.3 is that the choice of $\varepsilon > 0$ is arbitrary. In particular $\varepsilon > 0$ can be arbitrarily small. In fact, we ~~need~~ only need to care about small ε. Because if

$$|a_n - L| < \varepsilon_1 \quad \text{for } \varepsilon_1 < \varepsilon_2$$

then of course $|a_n - L| < \varepsilon_2$. Or if eventually the distance of a_n & L is less than a small ε_1, then of course they are less than any other ε_2 that is bigger than ε_1.

③ If $\{a_n\}$ does not converge anywhere as $n \to \infty$, then we say that $\{a_n\}$ is divergent.

④ Convergence or divergence of a sequence concerns the asymptotic behavior of the sequence $\{a_n\}_{n\geq 1}$. It's impossible for us to go over every terms to determine such behivor as the number of terms are infinitely many. Instead we use a logic lanague to describe it. In fact, there are symbols for logic terms:

for any : $\forall$

there exists : $\exists$

The we may refrase definition 1.3 as

"$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{Z}_+$ s.t. $|a_n - L| < \varepsilon$ $\forall n \geq n_0$."

Although, calculus were discovered by Newton in 17th century. The rigorous definition of convergence sequence were given by Bolzano & Weierstrauss in 19th century. Note all rigorous definitions of continuity, differentiability & integrability are based on the definition of convergent sequences.

In other words, people were using calculus without really understand what are calculus for about 200 years. ~~These~~ It partially explain why definition 1.3 is a difficult one.

To better understand definition, we for sure need to have some concrete examples. Here is the second principle of learning analysis or even mathematics:

(II). One should always use concrete examples to help them understand abstract notions or theorems.

But before we consider examples, we first note the following important facts concerning convergent sequence.

Definition 1.4. Let $\{a_n\}$ be a sequence. A sequence $\{b_k\}_{k\geq 1}$ is called a sub-sequence of $\{a_n\}$ if $b_1 = a_{n_1}$, $b_2 = a_{n_2}$, $\cdots$, $b_k = a_{n_k}$, $\cdots$ for a sequence of increasing positive integers

$$n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots$$

We write such a sequence $\{b_k\}_{k\geq 1}$ as $\{a_{n_k}\}_{k\geq 1}$.

Geometrically, it's not so hard to see that if $\lim_{n\to\infty} a_n = L$ then $\lim_{k\to\infty} b_k = L$ for all such sub-sequence. Because $\{b_k\} = \{a_{n_k}\}$ is part of $\{a_n\}$. So if a_n tends L as $n\to\infty$, then of course $b_k = a_{n_k}$ tends L as $k\to\infty$ (or $n_k\to\infty$). This ~~+~~ is the following theorem.

Theorem 1.2. If $\lim_{n\to\infty} a_n = L$, then $\lim_{k\to\infty} a_{n_k} = L$ for all subsequence $\{a_{n_k}\}$ of $\{a_n\}$.

Proof: How to put the geometric picture into rigorous math proof? Recall $\lim_{n\to\infty} a_n = L$ means $\forall \varepsilon > 0$, $\exists n_0$ s.t. $|a_n - L| < \varepsilon$, $\forall n \geq n_0$.

Now let's fix any subsequence $\{a_{n_k}\}$ of $\{a_n\}$. Then ~~+~~ there is a k_0 s.t. $n_{k_0} \geq n_0$. Since the sequence $\{n_k\}_{k\geq 1}$ increases in k, we have

$$n_k \geq n_{k_0} \geq n_0 \quad \text{for all } k \geq k_0$$

which implies $|a_{n_k} - L| < \varepsilon$. ~~for~~ $\forall k \geq k_0$

Thus we've just showed that :
$\forall \varepsilon > 0, \exists k_0$, s.t. $|a_{n_k} - L| < \varepsilon \quad \forall k \geq k_0$

By definition $\lim_{k \to \infty} a_{n_k} = L$. $\square$

<u>Theorem 1.3</u> (uniqueness of limit):
If $\lim_{n \to \infty} a_n = L$ & $\lim_{n \to \infty} a_n = M$, then $L = M$.
In other words, if $\{a_n\}$ is convergent, then they may only converges to an unique limit.

Idea of Proof: Geometrically, if a_n tends to L as $n \to \infty$, it's certainly not possible for a_n to tends to a different M as $n \to \infty$. How to put it into rigorous math proof? We argue by contradiction.

Proof: Assume $L \neq M$.
We may assume that $L < M$. The we can find a small ε s.t. $L + \varepsilon \leq M - \varepsilon$, e.g. any $\varepsilon \leq \frac{M-L}{2}$ will do.
Indeed, $L + \varepsilon \leq L + \frac{M-L}{2} = \frac{M+L}{2} = M - \frac{M-L}{2} \leq M - \varepsilon$
if $\varepsilon \leq \frac{M-L}{2}$.

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By definition $\lim_{n\to\infty} a_n = L$ means

$\forall \varepsilon > 0, \exists n_0$ s.t. $|a_n - L| < \varepsilon, \forall n \geq n_0$

In particular for $\varepsilon = \frac{M-L}{2}$, we can find such a n_0. Similarly for the same $\varepsilon = \frac{M-L}{2}$, we can find another $n_1 \in \mathbb{Z}_+$ s.t.

$$|a_n - M| < \varepsilon, \quad \forall n \geq n_1$$

since $\lim_{n\to\infty} a_n = M$ as well.

Thus for any $n \geq \max\{n_0, n_1\}$, it holds that

$$\begin{cases} |a_n - L| < \varepsilon \\ |a_n - M| < \varepsilon \end{cases}$$

$$\begin{aligned}
\Rightarrow |L-M| &\leq |L - a_n + a_n - M| \\
&\leq |L - a_n| + |a_n - M| \\
&< \varepsilon + \varepsilon \\
&= \frac{M-L}{2} + \frac{M-L}{2} \\
&= M-L \\
&= |M-L|
\end{aligned}$$

i.e. $|M-L| > |M-L|$, contradiction

Thus the assumption $M \neq L$ is false

$\Rightarrow M = L$ $\square$

$A \Longrightarrow B$: "A implies B"
$A \Longleftarrow B$: "B implies B"
$A \Longleftrightarrow B$: "A is equivalent to B"

1.2.2 Examples of convergent/divergent sequences.

Example 2.

① Fix $a>0$. Show that $\lim_{n\to\infty} \frac{1}{n^a} = 0$

Proof: By definition, we need to show that:

$\forall \varepsilon>0$, there is a n_0 s.t.

$$\left|\frac{1}{n^a} - 0\right| < \varepsilon, \quad \forall n \geq n_0.$$

The key thing we need to do here is to find the n_0.

How to do this? We work backwards:

$$\left|\frac{1}{n^a} - 0\right| < \varepsilon \iff \frac{1}{n^a} < \varepsilon \quad \left(\text{since } \frac{1}{n^a} > 0\right)$$

$$\iff \frac{1}{\varepsilon} < n^a$$

$$\iff \left(\frac{1}{\varepsilon}\right)^{\frac{1}{a}} < n.$$

Thus if we pick a n_0 s.t. $n_0 > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{a}}$, then by the argument above, we have for all $n \geq n_0 \left(> \left(\frac{1}{\varepsilon}\right)^{\frac{1}{a}}\right)$

$$\left|\frac{1}{n^a} - 0\right| < \varepsilon$$

To sum up, $\forall \varepsilon>0$, $\exists\ n_0 = \left\lceil \left(\frac{1}{\varepsilon}\right)^{\frac{1}{a}} \right\rceil$ s.t.

$$\left|\frac{1}{n^a} - 0\right| < \varepsilon, \quad \forall n \geq n_0$$

$\Rightarrow \lim_{n\to\infty} \frac{1}{n^a} = 0$ □

Corollary: $\lim\limits_{n\to\infty}\frac{1}{n}=0$, $\lim\limits_{n\to\infty}\frac{1}{n^2}=0$

$\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}=0$, $\lim\limits_{n\to\infty}\frac{1}{n^k}$ (for any $k\in\mathbb{Z}_+$).

② Fix $0<a<1$. Show that $\lim\limits_{n\to\infty} a^n = 0$

Proof: Here we need the formula of binomial expansion: $(x+y)^n=\sum\limits_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$

$$= x^n + n x^{n-1}\cdot y + \frac{n(n-1)}{2}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3\cdot 2\cdot 1}x^{n-3}\cdot y^3+\cdots+ n x y^{n-1} + y^n$$

$\binom{n}{k}=\frac{n\cdot(n-1)\cdots(n-k+1)}{k\cdot(k-1)\cdots 1}$

$\binom{n}{0}=\binom{n}{n}=1$

$\binom{n}{1}=\frac{n}{1}=n$

$\binom{n}{2}=\frac{n(n-1)}{2}$

Since $0<a<1 \Rightarrow \frac{1}{a}>1 \Rightarrow$ we may set $\frac{1}{a}=1+h$, where $h=\frac{1}{a}-1>0$

By definition, we need to show

$\forall \varepsilon>0, \exists n_0$ s.t. $|a^n-0|<\varepsilon \quad \forall n\geq n_0$.

Again the key thing here is to find n_0. We proceed by working backwards:

$$|a^n-0|<\varepsilon \Leftrightarrow a^n<\varepsilon \Leftrightarrow \frac{1}{\varepsilon}<\frac{1}{a^n}$$

$$\Leftrightarrow \frac{1}{\varepsilon}<\left(\frac{1}{a}\right)^n \Leftrightarrow \frac{1}{\varepsilon}<(1+h)^n$$

$$\Leftarrow \frac{1}{\varepsilon}<1+n\cdot h \quad \text{since } (1+h)^n=1+nh+\frac{n(n-1)}{2}h^2+\cdots>1+nh$$

$$\Leftrightarrow n>\frac{1}{h}\left(\frac{1}{\varepsilon}-1\right) \Leftrightarrow n>\left(\frac{1}{a}-1\right)\left(\frac{1}{\varepsilon}-1\right)$$

Since we only need to care about small ε, we may assume $\frac{1}{\varepsilon}-1>0$. Thus picking any

$$n_0 > \left(\frac{1}{a}-1\right)\left(\frac{1}{\varepsilon}-1\right), \text{ e.g. } \left\lceil\left(\frac{1}{a}-1\right)\left(\frac{1}{\varepsilon}-1\right)\right\rceil$$

By the argument above, we then have $\forall n \geq n_0$

$$|a^n - 0| < \varepsilon.$$

To sum up, $\forall \varepsilon>0$, $\exists n_0 = \left\lceil\left(\frac{1}{a}-1\right)\left(\frac{1}{\varepsilon}-1\right)\right\rceil$ s.t.

$$|a^n - 0| < \varepsilon, \quad \forall n \geq n_0$$

$\Rightarrow \lim_{n\to\infty} a^n = 0$. $\square$

Corollary: $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$, $\lim_{n\to\infty} e^{-n} = 0$, $\lim_{n\to\infty} \frac{1}{10^n} = 0$, ...

③ Fix $0<a<1$. Show that $\lim_{n\to\infty} n\cdot a^n = 0$

Proof: By definition, we need to show $\forall \varepsilon>0$, $\exists n_0$ s.t. $|n\cdot a^n - 0| < \varepsilon$ $\forall n \geq n_0$.

We work backwards to find the n_0

$$|n\cdot a^n - 0| < \varepsilon \Leftrightarrow n\cdot a^n < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < \frac{1}{n}\left(\frac{1}{a}\right)^n$$

$$\Leftrightarrow \frac{1}{\varepsilon} < \frac{1}{n}(1+h)^n \qquad \underline{\underline{h = \frac{1}{a}-1}}$$

$$\Leftarrow \frac{1}{\varepsilon} < \frac{1}{n}\,\frac{n(n-1)}{2}\cdot h^2 \qquad \text{since } (1+h)^n = 1 + nh + \frac{n(n-1)}{2}h^2 + \cdots > \frac{n(n-1)}{2}h^2$$

$$\Leftrightarrow \frac{1}{\varepsilon} < \frac{(n-1)}{2}\cdot h^2$$

$\Leftrightarrow n-1 > \frac{2}{\varepsilon h^2} \Leftrightarrow n > \frac{2}{\varepsilon h^2} + 1$

$\Leftrightarrow n > \frac{2a}{\varepsilon(1-a)} + 1$

Picking $n_0 = \lceil \frac{2a}{\varepsilon(1-a)} + 1 \rceil$, we have

$\forall n \geq n_0, \ |na^n - 0| < \varepsilon$.

To sum up, $\forall \varepsilon > 0, \exists n_0 = \lceil \frac{2a}{\varepsilon(1-a)} + 1 \rceil$ s.t.

$$|na^n - 0| < \varepsilon \quad \forall n \geq n_0$$

$\Rightarrow \lim_{n \to \infty} na^n = 0$ $\square$

Remark: Comparing examples ② & ③, we may see that the key thing here is that $\left(\frac{1}{a}\right)^n = (1+h)^n$ contains n, n^2 terms. In fact as $n \to \infty$, it contains n^k for any $k \in \mathbb{Z}$. Thus we may guess $\lim_{n \to \infty} n^k a^n = 0$. This is actually true. One of the basic facts in calculus or analysis: Exponential decaying like a^n $(0<a<1)$ always beats polynomial growth like n^k $(k \in \mathbb{Z}_+)$.

After going through some difficult examples, let's go back to two simple examples.

Example 2.

① Show that constant sequences are convergent & their limits are the single value in the range.

Proof: Let $\{a_n\}_{n\geq 1}$ be a constant sequence. Then $\exists\, c \in \mathbb{R}$ s.t. $a_n \equiv c$. We need to show by definition, i.e. $\forall \varepsilon > 0$, $\exists\, n_0$ s.t.

$$|a_n - c| < \varepsilon, \quad \forall n \geq n_0$$

Here since $a_n = c$ for all n, we obtain

$$|a_n - c| = 0 < \varepsilon \text{ for all } \varepsilon > 0 \text{ \& all } n \geq 1.$$

Thus, $\forall \varepsilon > 0$, $\exists\, n_0 = 1$ s.t.

$$|a_n - c| = 0 < \varepsilon \quad \forall n \geq n_0 = 1.$$

$\Rightarrow \lim a_n = c$.

Remark: Geometrically, it's kind of obvious since a_n stay still at c for all $n \geq 1$.

$\{a_n\}$
A sequence is called eventually constant if
$\exists\, c \in \mathbb{R}$, $\exists\, n_0 \in \mathbb{Z}_+$ s.t. $a_n = c \;\; \forall n \geq n_0$.

② Any eventually constant sequence $\{a_n\}$ is convergent & its limit is the value it takes on for all large n.

Proof: Similarly to ①. $\forall \varepsilon > 0$, $\exists\, n_0$ (the n_0 s.t. $a_n = c \;\; \forall n \geq n_0$) s.t.

$$|a_n - c| = 0 < \varepsilon, \; \forall n \geq n_0$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0.$$

Example 3. Divergent sequences.

① Let $a_n = \begin{cases} 1, & n \text{ even} \quad (\text{i.e. } n = 2k,\ k \in \mathbb{Z}_+) \\ -1, & n \text{ odd} \quad (\text{i.e. } n = 2k-1,\ k \in \mathbb{Z}_+). \end{cases}$

Show that it's divergent.

Proof: Theorem 1.2 says that if $\lim_{n\to\infty} a_n = L$, then $\lim_{k\to\infty} a_{n_k} = L$ for any subsequence $\{a_{n_k}\}$ of $\{a_n\}$.

Here we can find two subsequences

$$\{a_{n_k}\} = \{a_{2k}\}_{k\geq 1} = \{1\}_{k\geq 1} \Rightarrow \lim a_{2k} = 1$$

$$\{a_{n_k}\} = \{a_{2k-1}\}_{k\geq 1} = \{-1\}_{k\geq 1} \Rightarrow \lim a_{2k-1} = -1$$

& $1 \neq -1$. Thus $\{a_n\}$ is divergent since it has two convergent subsequences with different limits.

1.3 Operations with Limits

Theorem 1.4 (The Sum Law):
Let $\{a_n\}$ & $\{b_n\}$ be two convergent sequences. Then $\{a_n+b_n\}_{n\geq 1}$ is convergent &

$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n,$$

i.e. the limit of the sum is the sum of the limits.

Proof: We may assume $\lim_{n\to\infty} a_n = A$ & $\lim_{n\to\infty} b_n = B$.
Fix any $\varepsilon > 0$. Apply the definition of convergence with $\varepsilon/2$ for both $\{a_n\}$ & $\{b_n\}$, we may find a n_1 s.t. $|a_n - A| < \frac{\varepsilon}{2}$ $\forall n \geq n_1$;
and a n_2 s.t. $|b_n - B| < \frac{\varepsilon}{2}$ $\forall n \geq n_2$.
Set $n_0 = \max\{n_1, n_2\}$. Then $\forall n \geq n_0$, we have $|a_n - A| < \frac{\varepsilon}{2}$ & $|b_n - B| < \frac{\varepsilon}{2}$.
Thus $\forall n \geq n_0$ we have

$$\begin{aligned}|(a_n+b_n)-(A+B)| &= |(a_n-A)+(b_n-B)| \\ &\leq |a_n-A| + |b_n-B| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

i.e. $\forall \varepsilon > 0$, $\exists n_0$ s.t. $|(a_n+b_n)-(A+B)| < \varepsilon$ $\forall n \geq n_0$

$$\Rightarrow \lim_{n\to\infty}(a_n+b_n) = A+B = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \qquad \square$$

Corollary: Let $\{a_n\}$ be a convergent sequence & $c \in \mathbb{R}$ be a constant. Then

$$\lim_{n\to\infty} (a_n + c) = \lim_{n\to\infty} a_n + c$$

Proof: Taking $b_n \equiv c$. By Thm 1.4

$$\lim_{n\to\infty} (a_n+c) = \lim_{n\to\infty} (a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

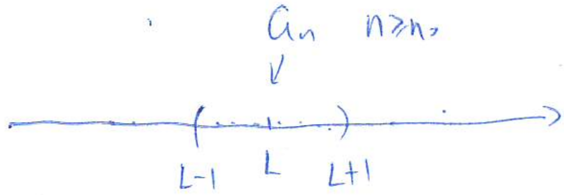
$$= \lim_{n\to\infty} a_n + c$$

Here we use $\lim_{n\to\infty} b_n = c$ $\square$

Definition 1.4. We say a sequence $\{a_n\}$ is bounded if $\exists M>0$ s.t.

$$|a_n| < M \quad \forall\ n \geq 1.$$

Remark: The boundedness of a sequence & of ~~that~~ a subset $S \subseteq \mathbb{R}$ is slightly different since $\{a_n\}_{n\geq 1}$ is a map & S is a set. However, $\{a_n\}_{n\geq 1}$ is bounded is equivalent to say that the range of $\{a_n\}_{n\geq 1}$ (as a subset of $\mathbb{R}$) is bounded. Indeed, if $\{a_n\}$ is bounded, then the range of $\{a_n\}$ is bounded by (below by $-M$ & above by M). Conversely.



if the range of $\{a_n\}_{n\geq 1}$ is bounded (below by m & above by M), then

$$m \leq a_n \leq M \quad \text{for all } n \geq 1$$

Then by taking $M_0 = \max\{|m|+1, |M|+1\}$, we for sure have $|a_n| < M_0 \ \forall\, n \geq 1$.

Theorem 1.5: Convergent sequences are bounded.

Proof: Let $\{a_n\}$ be a convergent sequence. ~~Then~~ We may assume $\lim\limits_{n\to\infty} a_n = L$. Then $\exists\, n_0$ s.t. $|a_n - L| < 1 \quad \forall\, n \geq n_0$.

By triangle inequality,

$$|a_n| - |L| \leq |a_n - L| < 1, \quad \forall n \geq n_0$$

$$\Rightarrow \ |a_n| < |L| + 1 \quad \forall\, n \geq n_0$$

In other words $\{a_n, n \geq n_0\}$ is bounded.

But $\{a_n, 1 \leq n \leq n_0 - 1\}$ is bounded since it's a finite set. Now

$$\{a_n, n \geq 1\} = \{a_n; 1 \leq n < n_0\} \cup \{a_n;\ n \geq n_0\}$$

is bounded since it's a union of two bounded sets $\Rightarrow$ $\{a_n\}_{n\geq 1}$ is bounded. $\square$

Example: Let $a_n = n$, $\forall n \geq 1$. Then $\{a_n\}$ is divergent.

Proof: $\{a_n\}_{n\geq 1}$ is unbounded since its range is $\mathbb{Z}_+$. Hence it cannot be convergent by Thm 1.5.

Theorem 1.6 (The product Law):
Let $\{a_n\}$ & $\{b_n\}$ be two convergent sequences. Then $\{a_n \cdot b_n\}_{n\geq 1}$ is convergent. Moreover
$$\lim_{n\to\infty}(a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right)\cdot\left(\lim_{n\to\infty} b_n\right),$$
i.e. the limit of the product is the product of the limits.

Proof: By Thm 1.5, $\{b_n\}_{n\geq 1}$ is bounded.
In particular, $\exists M>0$ s.t. $|b_n| < M$ $\forall n \geq 1$.
Fix any $\varepsilon > 0$. Assume $\lim_{n\to\infty} a_n = A$ & $\lim_{n\to\infty} b_n = B$.
Then $\lim_{n\to\infty} a_n = A$ implies for $\frac{\varepsilon}{2M}$, $\exists n_1$ s.t.
$$|a_n - A| < \frac{\varepsilon}{2M}, \quad \forall n \geq n_1 \quad ;$$
Similarly for $\frac{\varepsilon}{2(|A|+1)}$, $\exists n_2$ s.t.

$$|b_n - B| < \frac{\varepsilon}{2(|A|+1)} \qquad \forall n \geq n_2$$

Set $n_0 = \max\{n_1, n_2\}$. Then for $n \geq n_0$, we have

$$\begin{cases} |a_n - A| < \dfrac{\varepsilon}{2M} \\ |b_n - B| < \dfrac{\varepsilon}{2(|A|+1)} \end{cases}.$$

Then $\forall n \geq n_0$

$$\begin{aligned} |(a_n \cdot b_n) - A \cdot B| &= |a_n \cdot b_n - Ab_n + Ab_n - AB| \\ &\leq |a_n b_n - Ab_n| + |Ab_n - AB| \\ &= |b_n| \cdot |a_n - A| + |A| \cdot |b_n - B| \\ &< M \cdot \frac{\varepsilon}{2M} + |A| \cdot \frac{\varepsilon}{2(|A|+1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

To sum up, $\forall \varepsilon > 0$, $\exists n_0$ s.t.

$$|a_n \cdot b_n - AB| < \varepsilon \qquad \forall n \geq n_0$$

$$\Rightarrow \lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right) \qquad \square$$

Corollary. Let $\{a_n\}$ be a convergent sequence. Then $\lim_{n\to\infty} (c\,a_n) = c \lim_{n\to\infty} a_n$ for any constant $c \in \mathbb{R}$.

Theorem 1.7: Let $\{b_n\}$ be a convergent sequence with a nonzero limit. In other words, $\lim_{n\to\infty} b_n = B$ & $B \neq 0$. Then

$\exists n^*$ s.t. $|b_n| > \frac{|B|}{2}$ $\forall n \geq n$.

Proof: Taking $\varepsilon = \frac{|B|}{2} > 0$, then

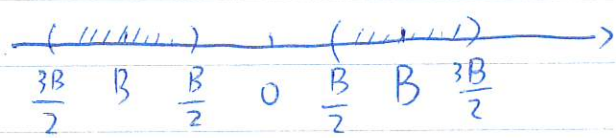
$\lim_{n\to\infty} b_n = B$ implies that $\exists n^*$ s.t.

$$|b_n - B| < \frac{|B|}{2}, \quad \forall n \geq n^*$$

By triangle inequality,

$$|B| - |b_n| < \frac{|B|}{2}, \quad \forall n \geq n^*$$

$$\Rightarrow |b_n| > |B| - \frac{|B|}{2} = \frac{|B|}{2}, \quad \forall n \geq n^* \qquad \square$$



Theorem 1.8 (The Quotient Law)

Let $\{a_n\}$ & $\{b_n\}$ be two convergent sequences & $\lim_{n\to\infty} b_n \neq 0$. Then

$\left\{\frac{a_n}{b_n}\right\}_{n\geq 1}$ converges & $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

i.e. the limit of the quotient is the qotient of the limits.

Proof: Since $\lim_{n\to\infty} b_n = B$ with $B \neq 0$, we may find a $n^* \in \mathbb{Z}_+$ s.t.

$$|b_n| > \frac{|B|}{2} \quad \forall n \geq n_*.$$

Fix any $\varepsilon > 0$. (Again we assume $\lim a_n = A$)

Then for $\frac{\varepsilon |B|}{4}$, we can find a n_1 s.t.

$$|a_n - A| < \frac{\varepsilon |B|}{4}, \quad \forall n \geq n_1;$$

for $\frac{|B|^2}{4(|A|+1)} \cdot \varepsilon$, we can find a n_2 s.t.

$$|b_n - B| < \frac{|B|^2 \cdot \varepsilon}{4(|A|+1)}, \quad \forall n \geq n_2$$

Set $n_0 = \max\{n^*, n_1, n_2\}$. Then $\forall n \geq n_0$ we have

$$\begin{cases} |b_n| > \frac{|B|}{2} \\ |a_n - A| < \frac{\varepsilon |B|}{4} \\ |b_n - B| < \frac{\varepsilon |B|^2}{4(|A|+1)} \end{cases}$$

Then we have $\forall n \geq n_0$ that:

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$$\left|\frac{a_n}{b_n}-\frac{A}{B}\right| = \left|\frac{a_nB - Ab_n}{b_n \cdot B}\right|$$

$$= \frac{|a_nB - AB + AB - Ab_n|}{|b_n| \cdot |B|}$$

$$< \frac{|B|\,|a_n - A| + |A| \cdot |B - b_n|}{\frac{|B|}{2} \cdot |B|}$$

$$< \frac{2}{|B|^2}\left(|B| \cdot \frac{\varepsilon|B|}{4} + |A| \cdot \frac{\varepsilon|B|^2}{4(|A|+1)}\right)$$

$$< \frac{2|B|^2}{|B|^2}\left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right)$$

$$= \varepsilon.$$

To sum up. $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{Z}_+$ s.t.

$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| < \varepsilon \qquad \forall n \geq n_0$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B} = \frac{\lim a_n}{\lim b_n} \qquad \square$$

Corollary: Let $\{a_n\}$ be a convergent sequence & $\lim a_n \neq 0$. Then $\lim_{n\to\infty}\left(\frac{c}{a_n}\right) = c/\lim_{n\to\infty} a_n$

* From now on, for a sequence $\{a_n\}$, if we say something holds true for all n sufficient larg, then it means $\exists\, n^* \in \mathbb{Z}_+$ s.t. it holds true for all $n \geq n^*$

Theorem 1.9 (The Squeeze Theorem).
Suppose $\{a_n\}$, $\{b_n\}$, & $\{c_n\}$ are sequences s.t. $a_n \leq b_n \leq c_n$ for all n sufficient large. More $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = A$. Then

$$\lim_{n\to\infty} b_n = A.$$

Proof: We may fix a n^* s.t.

$$a_n \leq b_n \leq c_n \quad \forall\, n \geq n^*. \qquad ①$$

~~By~~ Fix any $\varepsilon > 0$. By definition,

$\exists\, n_1$ s.t. $|a_n - A| < \varepsilon/3 \quad \forall\, n \geq n_1$; ②

$\exists\, n_2$ s.t. $|c_n - A| < \varepsilon/3 \quad \forall\, n \geq n_2$ ③

Set $n_0 = \max\{n^*, n_1, n_2\}$. Then we have ①, ②, & ③ $\forall\, n \geq n_0$. Then $\forall\, n \geq n_0$

it holds that

$$|b_n - A| = |b_n - c_n + c_n - A|$$

$$\le c_n - b_n + |c_n - A| \qquad c_n - b_n = |b_n - c_n| \text{ as } c_n \ge b_n \ \forall n \ge n_*$$

$$\le c_n - a_n + |c_n - A| \qquad b_n \ge a_n \ \forall n \ge n_*$$

$$= |c_n - a_n| + |c_n - A|$$

$$\le |c_n - A| + |a_n - A| + |c_n - A|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \qquad n \ge n_1, n \ge n_2$$

$$= \varepsilon$$

To sum up, $\forall \varepsilon > 0, \exists n_0$ s.t. $|b_n - A| < \varepsilon \ \forall n \ge n_0$

$\Rightarrow \lim_{n \to \infty} b_n = A$ $\square$

Theorem 1.10 Let $\{a_n\}$ & $\{b_n\}$ be two convergent sequences. Suppose $a_n \le b_n$ for all n sufficiently large. Then $\lim a_n \le \lim b_n$.

Proof: Let $A = \lim a_n$ & $B = \lim b_n$.

Fix n_* s.t. $a_n \le b_n \ \forall n \ge n_*$.

Find n_1 s.t. $|a_n - A| < \frac{\varepsilon}{2} \ \forall n \ge n_1$;

n_2 s.t. $|b_n - B| < \frac{\varepsilon}{2} \ \forall n \ge n_2$

Set $n_0 = \max\{n_*, n_1, n_2\}$. Then $\forall n \geq n_0$, we have

$$a_n \leq b_n \quad ①$$

$$A - a_n \leq |A - a_n| < \frac{\varepsilon}{2} \Rightarrow A < a_n + \frac{\varepsilon}{2} \quad ②$$

$$b_n - B \leq |b_n - B| < \frac{\varepsilon}{2} \Rightarrow b_n < B + \frac{\varepsilon}{2} \quad ③$$

$$\Rightarrow \forall n \geq n_0, \quad A \overset{②}{<} a_n + \frac{\varepsilon}{2} \overset{①}{\leq} b_n + \frac{\varepsilon}{2} \overset{③}{<} B + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = B + \varepsilon$$

$\Rightarrow A < B + \varepsilon, \ \forall \varepsilon > 0.$

$\Rightarrow A - B < \varepsilon, \ \forall \varepsilon > 0.$ This is only possible when $A - B \leq 0 \Rightarrow A \leq B \Rightarrow \lim a_n \leq \lim b_n$ □

Example 1. Compute $\lim\limits_{n\to\infty} \dfrac{n^4 + n^3 + 2n^2}{2n^4 + n^2 + 1}$

Solution: $$\lim_{n\to\infty} \frac{n^4+n^3+2n^2}{2n^4+n^2+1} = \lim_{n\to\infty} \frac{(n^4+n^3+2n^2)/n^4}{(2n^4+n^2+1)/n^4}$$

$$= \lim_{n\to\infty} \frac{1 + \frac{1}{n} + \frac{2}{n^2}}{2 + \frac{1}{n^2} + \frac{1}{n^4}}$$

Note $\lim\left(2 + \frac{1}{n^2} + \frac{1}{n^4}\right) \overset{S.L}{=} 2 + \lim \frac{1}{n^2} + \lim \frac{1}{n^4} = 2 \neq 0$

By Q.L. the limit above is

$$\overset{Q.L.}{=} \frac{\lim\left(1 + \frac{1}{n} + \frac{2}{n^2}\right)}{\lim\left(2 + \frac{1}{n^2} + \frac{1}{n^4}\right)} \overset{S.L.}{=} \frac{1 + \lim \frac{1}{n} + \overbrace{2\lim \frac{1}{n^2}}^{P.L.}}{2} = \frac{1}{2} \quad □$$

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Example 2. Show that $\left\{ a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \right\}$ is convergent & compute its limit.

Solution. Clearly for each $1 \le k \le n$, it holds that $\frac{1}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+k}} \le \frac{1}{\sqrt{n^2+1}}$. Thus

$$\underbrace{\frac{1}{\sqrt{n^2+n}} + \cdots + \frac{1}{\sqrt{n^2+n}}}_{n \text{ ~~times~~ terms}} \le a_n \le \underbrace{\frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+1}}}_{n \text{ terms}}$$

$$\Rightarrow \frac{n}{\sqrt{n^2+n}} \le a_n \le \frac{n}{\sqrt{n^2+1}}$$

But $\lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} \overset{Q.L.}{=\!=\!=} \frac{1}{\lim \sqrt{1+\frac{1}{n}}} = \frac{1}{1} = 1$

$1 \le \sqrt{1+\frac{1}{n}} \le 1+\frac{1}{n}$ & $\lim(1+\frac{1}{n}) = 1$

By Thm 1.9, $\lim \sqrt{1+\frac{1}{n}} = 1 \ne 0$

Similarly $\lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} \overset{Q.L.}{=\!=\!=} \frac{1}{\lim \sqrt{1+\frac{1}{n^2}}} = \frac{1}{1} = 1$

$1 \le \sqrt{1+\frac{1}{n^2}} \le 1+\frac{1}{n^2}$ & $\lim(1+\frac{1}{n^2}) = 1$

By Thm 1.9, $\lim \sqrt{1+\frac{1}{n^2}} = 1$

By Thm 1.9 again, $\lim\limits_{n}$ a_n converges & $\lim\limits_{n\to\infty} a_n = 1$ □

1.4 Monotone Sequences

Definition 1.5: Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers.

① $\{a_n\}$ is said to be monotone increasing if
$$a_n \leq a_{n+1} \quad \forall\, n \geq 1.$$

② $\{a_n\}$ is said to be monotone decreasing if
$$a_n \geq a_{n+1} \quad \forall\, n \geq 1.$$

③ A monotone sequence is a sequence that is either monotone increasing or monotone decreasing.

Remark: If the "inequality sign" ($\leq$ or $\geq$) becomes "strictly inequality sign" ($<$ or $>$), then we simply put "strictly" in front of monotone.

For instance, $\{a_n\}$ is strictly monotone increasing if $\quad a_n < a_{n+1} \quad \forall\, n \geq 1$.

First answer to the question: what is $\mathbb{R}$?

$\mathbb{R}$ is the set containing $\mathbb{Q}$ with a the <u>least-upper-bound property</u>. Specifically,

For each $S \subseteq \mathbb{R}$ that is bounded above, (i.e. $\exists M \in \mathbb{R}$ s.t. $x \leq M \ \forall x \in S$) there is a least-upper-bound α in the sense that: (i) α is an upper-bound of S (i.e. $x \leq \alpha \ \forall x \in S$); (ii) No number smaller than α is an upper-bound of S, i.e. $\forall \varepsilon>0, \exists x_0 \in S$ s.t.

$$x_0 > \alpha - \varepsilon$$

denoted $\alpha = \sup S$

(No matter how small ε is, there is some number in S that is strictly bigger than $\alpha - \varepsilon$) Such an α is called the least upper bound or supremum of S.

In some sense, the least-upper-bound property of $\mathbb{R}$ is not something that can be proved. It is the property defining what is $\mathbb{R}$.

Theorem 1.11: If a monotone increasing sequence is bounded above, then it's convergent.

Proof: Let $\{a_n\}$ be a monotone increasing sequence that is bounded above. So the range $\{a_n, n\geq 1\}$ is bound above. By the least-upper-bound property of $\mathbb{R}$, we may set

$$A = \sup\{a_n, n\geq 1\}.$$

i.e. A is the least-upper-bound ~~p~~ or supremum of $\{a_n, n\geq 1\}$. Then,

(i) $a_n \leq A$, $\forall n \geq 1$

(ii) $\forall \varepsilon > 0$, $\exists n_0$ s.t. $a_{n_0} > A - \varepsilon$.

But $\{a_n\}$ is monotone increasing, (ii) implies that $\forall n \geq n_0$, $a_n \geq a_{n_0} > A - \varepsilon$. Thus we may restate (ii) as:

$x \in (A-\varepsilon, A+\varepsilon) \Leftrightarrow |x-A| < \varepsilon$

$x \in (A-\varepsilon, A] \Rightarrow x \in (A-\varepsilon, A+\varepsilon) \Rightarrow |x-A| < \varepsilon$

(ii)' $\forall \varepsilon > 0, \exists n_0$ s.t. $a_n > A - \varepsilon \quad \forall n \geq n_0$

Combining (i) & (ii)', we obtain:

$\forall \varepsilon > 0, \exists n_0$ s.t.

$$A - \varepsilon < a_n \leq A \qquad \forall n \geq n_0$$

$$\Rightarrow |a_n - A| < \varepsilon, \quad \forall n \geq n_0$$

$\Rightarrow \lim_{n \to \infty} a_n = A.$ $\square$

Corollary: If $\{a_n\}$ is monotone decreasing & bound below, then it's convergent.

Proof: We claim $\{-a_n\}$ is monotone increasing & bounded above. Indeed,

(i) $a_n \geq a_{n+1} \ \forall n \geq 1 \Rightarrow -a_n \leq -a_{n+1} \quad \forall n \geq 1$

(ii) $a_n \geq m, \forall n \geq 1 \Rightarrow -a_n \leq -m \quad \forall n \geq 1$

as desired. Thus we can apply Thm 1.11 to $\{-a_n\}$ & obtain that

$$\lim_{n \to \infty} (-a_n) \quad \text{exists}.$$

By product law:

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} -1 \cdot (-a_n) = -\lim_{n \to \infty} (-a_n)$$

In particular, $\{a_n\}$ converges. $\square$

Theorem 1.12: The sequence

$$a_n = \left(1+\frac{1}{n}\right)^n$$

is monotone increasing & $2 < a_n < 3$, $\forall\, n \geq 1$.
In particular, it's convergent.

Proof: By binomial expansion,

$$a_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \cdot \left(\frac{1}{n}\right)^k = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}$$

For each $0 \leq k \leq n$, $\binom{n}{k}\frac{1}{n^k} = \dfrac{n(n-1)\cdots(n-k+1)}{k!\cdot n^k}$

$$= \frac{1}{k!}\,\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-k+1}{n} = \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

Thus $a_n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+$

$$\frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right). \qquad (*)$$

It implies that: $a_{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)$

$$+\cdots+\frac{1}{n!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right).$$

Clearly, $\frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) < \frac{1}{k!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right)$, $\forall\, 2 \leq k \leq n$

$\Rightarrow$ The kth term of a_{n+1} > The kth term of a_n.

$\Rightarrow a_{n+1} > a_n \quad \forall\, n \geq 1$

$\Rightarrow \{a_n\}$ is strictly monotone increasing.

Also, by the formula (*), we can see that

$2 < a_n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$

Clearly, $n! = n(n-1)\cdots 2\cdot 1 \geq \underbrace{2\cdot 2\cdots 2}_{n-1} = 2^{n-1} \quad \forall n\geq 1.$

$\Rightarrow \frac{1}{n!} \leq \frac{1}{2^{n-1}}, \quad \forall n \geq 1.$

$\Rightarrow \quad 2 < a_n < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$

But $1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} = \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = 2-(\frac{1}{2})^{n-1}$

$\Rightarrow 2 < a_n < 1+2-(\frac{1}{2})^{n-1} = 3-(\frac{1}{2})^{n-1} < 3 \qquad \forall n \geq 1.$

Since $\{a_n\}$ is monotone increasing & bounded above (by 3), it's convergent by Thm 1.11 $\square$

We denote $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(1+\frac{1}{n})^n$ by e, i.e.

$$e = \lim_{n\to\infty}(1+\frac{1}{n})^n.$$

Example 1: Compute $\lim_{n\to\infty}(1+\frac{1}{2n})^n$ & $\lim_{n\to\infty}(1+\frac{1}{n})^{\frac{n}{2}}$

Solution. ① $\{b_n = (1+\frac{1}{2n})^{2n}\}$ is a subsequence of $\{a_n = (1+\frac{1}{n})^n\}$. So it's monotone increasing & bounded above. Thus $\{\sqrt{b_n} = (1+\frac{1}{2n})^n\}$ is monotone increasing & bounded above. By Thm 1.11 $\lim_{n\to\infty}(1+\frac{1}{2n})^n$ exists

By product law $\lim_{n\to\infty}(1+\frac{1}{2n})^n \cdot \lim_{n\to\infty}(1+\frac{1}{2n})^n$

$= \lim_{n\to\infty}\left[(1+\frac{1}{2n})^n \cdot (1+\frac{1}{2n})^n\right] = \lim_{n\to\infty}(1+\frac{1}{2n})^{2n} = \lim_{n\to\infty} b_n = e$

since $\{b_n\}$ is a subsequence of $\{a_n\}$

$\Rightarrow \left(\lim_{n\to\infty}(1+\frac{1}{2n})^n\right)^2 = e \Rightarrow \lim_{n\to\infty}(1+\frac{1}{2n})^n = \sqrt{e}$

② Since $\{a_n = (1+\frac{1}{n})^n\}$ is monotone increasing & bounded above $\Rightarrow \{\sqrt{a_n} = (1+\frac{1}{n})^{\frac{n}{2}}\}$ is monotone increasing & bounded above $\Rightarrow \lim_{n\to\infty}(1+\frac{1}{n})^{\frac{n}{2}}$ exists

$\Rightarrow \lim_{n\to\infty}(1+\frac{1}{n})^{\frac{n}{2}} \lim_{n\to\infty}(1+\frac{1}{n})^{\frac{n}{2}} = \lim_{n\to\infty}(1+\frac{1}{n})^{\frac{n}{2}} \cdot (1+\frac{1}{n})^{\frac{n}{2}}$

$= \lim_{n\to\infty}(1+\frac{1}{n})^n = e$

$\Rightarrow \left(\lim_{n\to\infty}(1+\frac{1}{n})^{\frac{n}{2}}\right)^2 = e \Rightarrow \lim_{n\to\infty}(1+\frac{1}{n})^{\frac{n}{2}} = \sqrt{e}$ $\square$

Definition 1.6: We say $\lim_{n\to\infty} a_n = \infty$ if $\forall N \in \mathbb{Z}_+, \exists n_0$ s.t. $a_n \geq N$, $\forall n \geq n_0$. (Note such $\{a_n\}$ is divergent)

Theorem 1.13: Let $\{a_n\}$ be monotone increasing & unbounded. Then $\lim_{n\to\infty} a_n = \infty$.

Proof: Since $\{a_n\}$ is unbounded, $\forall N \in \mathbb{Z}_+$, $\exists n_0$ s.t. $a_{n_0} \geq N \Rightarrow \forall n \geq n_0$, $a_n \geq a_{n_0} \geq N$

$\Rightarrow \lim_{n\to\infty} a_n = +\infty$.

* Similarly, one can show that if $\{a_n\}$ is monotone decreasing & unbounded, then $\lim_{n\to\infty} a_n = -\infty$

1.5 Bolzano-Weierstrass Theorem

Defition 1.6: Let $S \subseteq \mathbb{R}$ be a subset. We say $\alpha \in \mathbb{R}$ is a limit point of S if:

$$\forall \varepsilon > 0, \ \exists\, a \in S \setminus \{\alpha\} \ \text{ s.t.}$$

$$|a - \alpha| < \varepsilon.$$

We may also call such an α a cluster point or a point of accumulation of S.

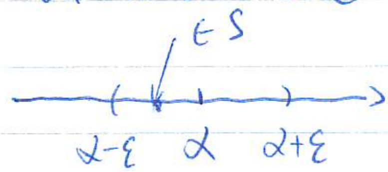
Remark: Note that the notion of limit pt describe the relation between a subset S of $\mathbb{R}$ & a point $\alpha \in \mathbb{R}$. Here α might or might not belong to S.

The following theorem tells us the relation between a limit point of a set & the limit of a convergent sequence.

Theorem 1.14. If α is a limit point of S, then there exists a sequence $\{a_n\}$ of mutually distinct points that belong to S s.t.

$$\lim_{n \to \infty} a_n = \alpha.$$

Geometrically it's not so hard to visualize it. The definition 1.6 basically says that no matter how small $\varepsilon > 0$ is, the open interval centered at α always contains at least one point of $S \setminus \{\alpha\}$.



In particular, for each n if we set $\varepsilon = \frac{1}{n}$, then we will find a point $a_n \in S\setminus\{\alpha\}$ s.t. $|a_n - \alpha| < \frac{1}{n}$. We can then show $\lim a_n = \alpha$.
But we need to make sure that a_n's are mutually different, i.e. $a_n \neq a_k \ \forall\, n \neq k$

Proof: By definition 1.6, for $\varepsilon_1 = 1$, we can find a ~~$a_1 \neq \alpha$ s.t.~~ $a_1 \in S \setminus \{\alpha\}$ s.t.

$$|a_1 - \alpha| < 1$$

Then we set $\varepsilon_2 = \min\{\frac{1}{2}, |a_1 - \alpha|\} > 0$. For ε_2, we can find a $a_2 \in S\setminus\{\alpha\}$ s.t.

$$|a_2 - \alpha| < \varepsilon_2$$

Note $|a_2 - \alpha| < \varepsilon_2 \le \frac{1}{2}$ & $|a_2 - \alpha| < \varepsilon_2 \le |a_1 - \alpha|$ which implies that $a_2 \neq a_1$.

Proceeds like this. We may then obtain for each $n \in \mathbb{Z}^+$ & each

$$\varepsilon_n = \min\{\tfrac{1}{n}, |a_1 - \alpha|, \dots, |a_{n-1} - \alpha|\} > 0$$

a point $a_n \in S\setminus\{\alpha\}$ s.t.

$$|a_n - \alpha| < \varepsilon_n$$

Note that $|a_n - \alpha| < \varepsilon_n \le \frac{1}{n}$ &

$|a_n - \alpha| < \varepsilon_n \le |a_k - \alpha| \quad \forall\, k = 1, \cdots, n-1$ which implies that $a_n \ne a_1, a_n \ne a_2, \cdots, a_n \ne a_{n-1}$.

Thus we obtain a sequence of mutually different points $\{a_n\}_{n\ge1}$ in $S\setminus\{\alpha\}$ s.t.

$$|a_n - \alpha| < \frac{1}{n}$$

$$\Rightarrow \quad -\frac{1}{n} < a_n - \alpha < \frac{1}{n}$$

But $\lim\limits_{n\to\infty}\left(-\frac{1}{n}\right) = \lim\limits_{n\to\infty}\left(\frac{1}{n}\right) = 0$. By squeeze thm

$$\lim_{n\to\infty}(a_n - \alpha) = 0$$

By sum & product laws: $a_n = a_n - \alpha + \alpha$ conv.

$\cancel{0 = \lim\limits_{n\to\infty}(a_n - \alpha) = \lim\limits_{n\to\infty}}$ & its limit is:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(a_n - \alpha + \alpha) = \lim_{n\to\infty}(a_n - \alpha) + \alpha$$

$$= \alpha. \qquad \square$$

Remark: Obviously, the converse statement of Thm 1.14 is true as well. Indeed, if $\exists \{a_n\}_{n\geq 1}$ s.t. $a_n \neq a_k$ $\forall n, k \in \mathbb{Z}_+$ & $a_n \in S\setminus\{\alpha\}$

& $\lim_{n\to\infty} a_n = \alpha$.

Then $\forall \varepsilon > 0$, $\exists n_0$ s.t. $|a_n - \alpha| < \varepsilon$, $\forall n \geq n_0$.

In particular, there is a $a_n \in S\setminus\{\alpha\}$ s.t.

$$|a_n - \alpha| < \varepsilon.$$

$\Rightarrow$ ~~a~~ α is a limit pt of S. Thus Thm 1.14 is actually "if & only if"

Corollary: A finit set S cannot have limit pt.

Proof: By Thm 1.14, if S has a limit pt α, then ~~&~~ S contains a sequence of mutually distinct pts $\{a\}_{n\geq 1}$. Clearly, card$\{a_n, n\geq 1\}$ is infinite. But $\{a_n, n\geq 1\} \subseteq S$ which implies that Card(S) is infinite. In other words, we showed that if a set S has a limit pt, then its cardinality must be infinite $\Rightarrow$ finite set cannot have limit pts $\square$

Definition 1.7: We define S' to be the set of limit points of S, i.e. $\alpha \in S'$ if & only if α is a limit point of S. We define $\bar{S} := S \cup S'$ to be the closure of S. If $S' \subseteq S$, or equivalently $\bar{S} = S$, then we say S is a closed set.

Example 1

① A finite set S is closed.

Solution: Let S be a finite set. Then by the corollary of Thm 1.14, $S' = \emptyset \Rightarrow S' \subset S$ $\Rightarrow S$ is closed.

② A closed interval $I = [a,b]$ is closed.

Solution: We need to show $I' \subset I$. Equivalently, we show $\forall x \notin I$, $x \notin I'$ (i.e. $I^c \subset (I')^c$). If $x \notin I$, then either $x < a$ or $x > b$. Say $x < a$. Then we set $\varepsilon = a - x > 0$, & consider the interval $(x-\varepsilon, x+\varepsilon)$. For all $y \in (x-\varepsilon, x+\varepsilon)$, we have

$y < x + \varepsilon = x + (a - x) = a \Rightarrow y \notin I = [a, b]$

$\Rightarrow (x-\varepsilon, x+\varepsilon) \cap I = \emptyset$. In other words, for any $x < a$, we can find an $\varepsilon > 0$ s.t. there is no point z in I with $|z - x| < \varepsilon$.

$\Rightarrow x \notin I'$. Similarly, we can show if ~~y~~ $x > b$, then ~~y~~ $x \notin I'$. $\Rightarrow x \notin I'$ if $x \notin I$

$\Rightarrow I' \subset I$. $\square$

③. An open interval $I = (a, b)$ is not closed.

Solution: In fact, we can show $a \in I'$. But clearly $a \notin I$. Thus $I' \not\subset I \Rightarrow I$ is not closed.

$\forall \varepsilon > 0$, it clearly holds that

$$(a-\varepsilon, a+\varepsilon) \cap (a, b) = \begin{cases} (a, a+\varepsilon), & \text{if } a+\varepsilon \le b \\ (a, b) & \text{if } a+\varepsilon > b \end{cases}$$

$\neq \emptyset$. $\Rightarrow \exists\, x \in (a-\varepsilon, a+\varepsilon) \cap I$

$\Rightarrow |x - a| < \varepsilon$, $x \in I$ (and $x \neq a$ since $a \notin I$).

$\Rightarrow a$ is a limit point of I

& $a \notin I \Rightarrow I$ is not closed.

④ $S=\{\frac{1}{n};\ n\in\mathbb{Z}_+\}$ is not closed. (In fact $\bar{S}=\{0, \frac{1}{n};\ n\in\mathbb{Z}_+\}$)

Solution: clearly $\{a_n=\frac{1}{n}\}_{n\geq 1}$ is a sequence of mutually distinct points of S &

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$$

By the remark following Thm 1.14, $0\in S'$

But $0\notin S$. Thus S is not closed.

Definition 1.8: Let $\{I_n\}_{n\geq 1}$ be a sequence of closed intervals s.t.

$$I_1\supset I_2\supset I_3\supset\cdots\supset I_n\supset I_{n+1}\supset\cdots$$

Let λ_n be the length of I_n. We say $\{I_n\}_{n\geq 1}$ is a nest if in addition $\lim_{n\to\infty}\lambda_n=0$.

Theorem 1.15. Let $\{I_n\}_{n\geq 1}$ be a nest. Then there is a $\xi\in\mathbb{R}$ s.t. $\{\xi\}=\bigcap_{n=1}^{\infty} I_n$, i.e. the intersection of I_n, $n\geq 1$ is a single point set.

Proof: We write $I_n=[a_n, b_n]$ $n\geq 1$.

Since $I_n = [a_n, b_n] \supseteq [a_{n+1}, b_{n+1}] = I_n$

$\Rightarrow a_n \le a_{n+1} \le b_{n+1} \le b_n \quad \forall\, n \ge 1.$

In particular, $\{a_n\}_{n\ge 1}$ is a monotone increasing sequence that is bounded above (e.g. by b_1, or any b_n). By Theorem 1.11 $\{a_n\}$ is convergent, we may set

$$\lim_{n\to\infty} a_n = \xi.$$

Similarly, $\{b_n\}$ is a monotone decreasing sequence that is bounded below. Thus $\exists\, \eta \in \mathbb{R}$ s.t. $\lim\limits_{n\to\infty} b_n = \eta$.

By Theorem 1.10, $a_n \le b_n \quad \forall\, n \ge 1$

$$\Rightarrow \xi \le \eta$$

Recall from the proof of Thm 1.11. ξ is the $\sup\{a_n\} \Rightarrow a_n \le \xi \quad \forall\, n \ge 1.$

Similarly, $\eta = \inf\{b_n\}$, the greatest-lower-bound or the infimum of $\{b_n\}$. $\Rightarrow \eta \le b_n \quad \forall\, n \ge 1$

$\Rightarrow a_n \le \xi \le \eta \le b_n \quad \forall\, n \ge 1$

$\Rightarrow \xi \in I_n \;\forall\, n \ge 1 \Rightarrow \xi \in \bigcap\limits_{n\ge 1} I_n.$

Next, we need to show ξ is the only point in $\bigcap_{n=1}^{\infty} I_n$. Suppose $\boxed{\text{for the sake of contradiction}}$, there is $\xi' \in \bigcap_{n=1}^{\infty} I_n$, $\boxed{\xi' \neq \xi}$ Then

$$a_n \le \xi' \le b_n \quad \forall n \ge 1$$

$$\& \quad a_n \le \xi \le b_n \quad \forall n \ge 1$$

$$\Rightarrow 0 < |\xi - \xi'| \le |b_n - a_n| = \lambda_n \quad \forall n \ge 1$$

But $\lim_{n \to \infty} \lambda_n = 0$. By Squeeze Thm,

$$|\xi - \xi'| = 0$$

$\Rightarrow \xi' = \xi$. $\Rightarrow$ $\bigcap_{n=1}^{\infty} I_n$ contains only ξ, i.e.

$$\bigcap_{n=1}^{\infty} I_n = \{\xi\} \qquad \square$$

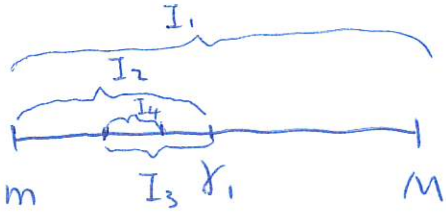
Theorem 1.16 (Bolzano-Weierstrauss Thm for sets)

Every bounded, infinite set of real numbers has at least one limit point.

Proof: Let S be such a bound & infinite set. Then $\exists$ $m \le M$ s.t.

$$m \le x \le M, \quad \forall x \in S, \text{ or } S \subseteq [m, M]$$

We denote $[m, M]$ by I_1.



Denote by γ_1 the midpoint $\frac{m+M}{2}$ of I_1 & consider two closed intervals

$[m, \gamma_1]$ & $[\gamma_1, M]$

Then at least one of them contains an infinite number of points in S.

Denote that interval by I_2.

Then we can repeat the same process we did on I_1 on I_2 & obtain a ~~a half~~ closed interval I_3 that contains infinite number of points of S. & I_3 is basicall a half of I_2.

Proceed by induction, we can then obtain a sequence of closed intervals $\{I_n\}_{n\geq 1}$ s.t.

① $I_n \supset I_{n+1}$, $\forall n \geq 1$

② $|I_n| = \frac{M-m}{2^{n-1}} \Rightarrow \lim |I_n| = \lim \frac{M-m}{2^{n-1}} = 2(M-m)\lim \frac{1}{2^n} = 0$

③ I_n contains an infinite number of pts in S

$\Rightarrow \{I_n\}_{n\geq 1}$ is a nest $\Rightarrow \exists \xi$ s.t.

$\{\xi\} = \bigcap_{n=1}^{\infty} I_n$. Now $\forall \varepsilon > 0$, $\exists n$ s.t. $\frac{M-m}{2^{n-1}} < \varepsilon$

Since $\xi \in I_n$, $|I_n| = \frac{M-m}{2^{n-1}} < \varepsilon$ & I_n contains infinite number of points in S, we may in particular pick a point $x \in I_n$ $\ni$ $x \in S$, $x \neq \xi$

Since $x, \xi \in I_n \Rightarrow |x-\xi| < |I_n| < \varepsilon$.

To summarize, $\forall \varepsilon > 0$, $\exists\, x \in S \setminus \{\xi\}$ s.t.

$$|x-\xi| < \varepsilon$$

$\Rightarrow \xi \in S'$ $\square$

Theorem (1.17) : (Balzano-Weierstrass Thm for sequences).

Every bounded sequence of real numbers has at least one convergent subsequence.

proof: Let $\{a_n\}$ be a bounded sequence. We divide it into two different cases.

Case I: The range of $\{a_n\}$ is a finite set.

Then at least one of points in the range $\{a_n, n \geq 1\}$ must occur infinitely number of times in the sequence $\{a_n\}$. otherwise, the sequence is finite since each pt in its range occurs in the sequence finite number of times.

Specifically, there is a $\xi \in \{a_n, n \geq 1\}$ s.t.

$\exists\ n_1 < n_2 < \cdots < n_k < \cdots$ of $\mathbb{Z}_+$ s.t.

$$a_{n_k} = \xi \quad \forall\ k \geq 1$$

$\Rightarrow \lim_{k \to \infty} a_{n_k} = \xi$ since the subsequence $\{a_{n_k}\}$ is a constant sequence.

Case II: The range $S = \{a_n ; n \geq 1\}$ is an infinite set.

Then by Thm 1.16, S has a limit pt ξ.

By definition, there is a $a_{n_1} \in S \setminus \{\xi\}$ s.t.

$$|a_{n_1} - \xi| < 1$$

Set $\underbrace{\varepsilon_2 = \min\{\tfrac{1}{2}, |a_1 - \xi|, \ldots, |a_{n_1} - \xi|, a_n \neq \xi\ 1 \leq n \leq n_1\}}_{>0}$.

Then we may find n_2 with $a_{n_2} \in S \setminus \{\xi\}$ s.t.

$$|a_{n_2} - \xi| < \varepsilon_2$$

Since $|a_{n_2} - \xi| \neq |a_n - \xi|$, $\forall\ 1 \leq n \leq n_1 \Rightarrow n_2 > n_1$.

Proceed by induction, we find a sequence $\{a_{n_k}\}_{k \geq 1}$ with $n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots$

s.t. $\quad |a_{n_k} - \xi| < \varepsilon_k \leq \frac{1}{k} \quad \forall\ k \geq 1.$

$\Rightarrow \{a_{n_k}\}$ is a subsequence of $\{a_n\}$ &

$$-\frac{1}{k} < a_{n_k} - \xi < \frac{1}{k} \Rightarrow \lim_{n \to \infty} a_{n_k} = \xi$$

$\square$

Chapter 2. Continuous Functions

2.1 Definition of continuous functions

2.1.1. Definitions of continuity

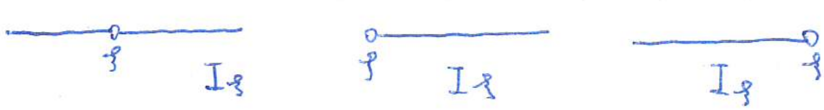
① Let $I \subseteq \mathbb{R}$ be an interval. Then we have:

- I is closed, i.e. $I = [a,b]$, $(-\infty, a]$, $[b, +\infty)$, or $\mathbb{R}$
- I is open, i.e. $I = (a,b)$, $(-\infty, a)$, $(b, +\infty)$, or $\mathbb{R}$
- I is half-open, i.e. $I = (a,b]$ or $[a,b)$.

② We say $x_0 \in I$ is an interior point of I if it's not an end-point of I. Note open intervals do not have end-points.

③ An open interval, e.g. $I = (a,b)$, containing x_0 is called a neighborhood of x_0. In particular, the open interval $(x_0 - \delta, x_0 + \delta)$ is called a δ-neighborhood of x_0. Note that $x \in (x_0 - \delta, x_0 + \delta)$
$$\Leftrightarrow |x - x_0| < \delta.$$

④ Let I be an interval & $\xi \in I$. We define
$$I_\xi = I \setminus \{\xi\}.$$
In other words, we remove ξ from I to obtain I_ξ.



Definition 2.1: Let I be an interval & $\xi \in I$. Let f be a function defined on $I_\xi = I \setminus \{\xi\}$. Suppose there is a $A \in \mathbb{R}$ s.t.

$\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $|f(x) - A| < \varepsilon$ $\forall x$ with $|x - \xi| < \delta$ & $x \in I_\xi$ (or equivalently $x \in (\xi - \delta, \xi + \delta) \cap I_\xi$).

Then we say $f(x)$ converges to A (tends to A) as x converges to ξ (tends to ξ) and write

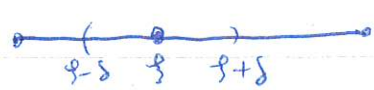
$$\lim_{x \to \xi} f(x) = A \quad \text{or}$$

$$f(x) \to A \text{ as } x \to \xi.$$

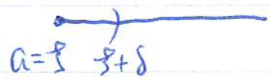
Remark ①: Note $x \in I_\xi$ implies that $x \neq \xi$. Indeed, we only assume that f is defined on I_ξ. Thus f might not defined at ξ, i.e. there might not be a $f(\xi)$.

② In the definition 2.1, if $I = [a, b]$ then there are three different cases:

(i) ξ is an interior point of I, i.e. $\xi \in (a, b)$. Then for δ very small, e.g. $\delta \leq \min\{|\xi - a|, |\xi - b|\}$, $(\xi - \delta, \xi + \delta) \subseteq I$. Thus $(\xi - \delta, \xi + \delta) \cap I_\xi = (\xi - \delta, \xi) \cup (\xi, \xi + \delta)$



In particular, $f(x) \to A$ as x tends to ξ both from left and from right.

(ii). If $\xi = a$, then 

$(\xi-\delta, \xi+\delta) \cap I_\xi = (\xi, \xi+\delta)$

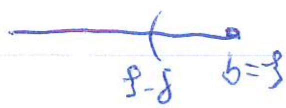
Thus in the definition, we only need: $\forall \varepsilon>0, \exists \delta>0$ s.t. $|f(x)-A| < \varepsilon, \ \forall x \in (\xi, \xi+\delta)$.

In other words, $f(x) \to A$ as $x \to \xi$ from right.

In this case, we write:

$$\lim_{x \to \xi+} f(x) = A \quad \text{or} \quad f(\xi+) = \lim_{x \to \xi+} f(x) = A.$$

or $f(x) \to A$ as $x \to \xi+$

(iii). If $\xi = b$, then similarly 

$(\xi-\delta, \xi+\delta) \cap I_\xi = (\xi-\delta, \xi)$

Then $\forall \varepsilon>0, \exists \delta>0$. s.t. $|f(x)-A| < \delta$. $\forall x \in (\xi-\delta, \xi)$.

In this case, $f(x) \to A$ as $x \to \xi$ from left.

We write $\lim\limits_{x \to \xi-} f(x) = f(\xi-) = A$

or $f(x) \to A$ as $x \to \xi-$

(iv). In fact, even if ξ is an interior pt of I. We can still define $\lim\limits_{x\to\xi+} f(x) = f(\xi+) = A$

if ~~as~~ : $\forall \varepsilon>0, \exists \delta>0$. s.t.

$$|f(x)-A|<\varepsilon, \quad \forall x\in(\xi, \xi+\delta)\cap I.$$

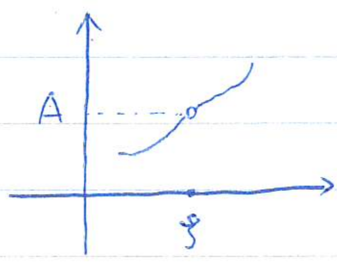
And we call A is the right limit of f at ξ.

Similarly, we define ~~f~~ $\lim\limits_{x\to\xi-} f(x) = f(\xi-) = A$

if : $\forall \varepsilon>0, \exists \delta>0$. s.t.

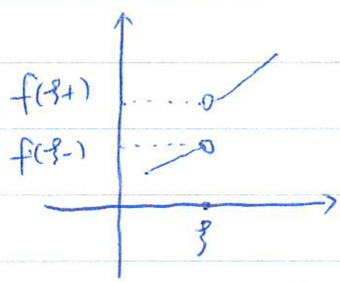
$$|f(x)-A|<\varepsilon, \quad \forall x\in(\xi-\delta, \xi)\cap I.$$

And we call A is the left limit of f at ξ.



$\lim\limits_{x\to\xi} f(x) = A$

$= f(\xi-) = f(\xi+)$



$\lim\limits_{x\to\xi+} f(\xi) = f(\xi+)$

$\neq f(\xi-) = \lim\limits_{x\to\xi-} f(x)$

Theorem 2.1. Let $\xi\in I$ be an interior point of I.
Then $\lim\limits_{x\to\xi} f(x) = A$ if and only if

$$\lim_{x\to\xi-} f(x) = \lim_{x\to\xi+} f(x) = A.$$

Proof: (i) First we show

$\lim_{x\to\xi} f(x) = A \Rightarrow \lim_{x\to\xi-} f(x) = \lim_{x\to\xi+} f(x) = A$.

By definition of $\lim_{x\to\xi} f(x) = A$, we have

$\forall \varepsilon>0, \exists \delta>0$, s.t. $|f(x)-A|<\delta$, $\forall x\in(\xi-\delta, \xi+\delta)\cap I_\xi$

In particular, $\forall \varepsilon>0, \exists\delta>0$. s.t.

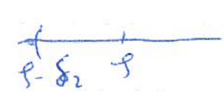
$|f(x)-A|<\delta$, $\forall$ $x\in(\xi, \xi+\delta)\cap I \Rightarrow \lim_{x\to\xi+} f(x) = A$
and
$x\in(\xi-\delta, \xi)\cap I \Rightarrow \lim_{x\to\xi-} f(x) = A$

(ii) Second, we want to show

$\lim_{x\to\xi-} f(x) = \lim_{x\to\xi+} f(x) = A \Rightarrow \lim_{x\to\xi} f(x) = A$.

By definitions of right & left limits, we have
$\forall \varepsilon>0$,
$\exists \delta_1>0$ s.t. $|f(x)-A|<\varepsilon$, $\forall x\in(\xi, \xi+\delta_1)\cap I$

$\exists\delta_2>0$ s.t. $|f(x)-A|<\varepsilon$, $\forall x\in(\xi-\delta_2, \xi)\cap I$



Set $\delta = \min\{\delta_1, \delta_2\}$. Then $\forall\varepsilon>0$, $\exists\delta>0$ s.t.

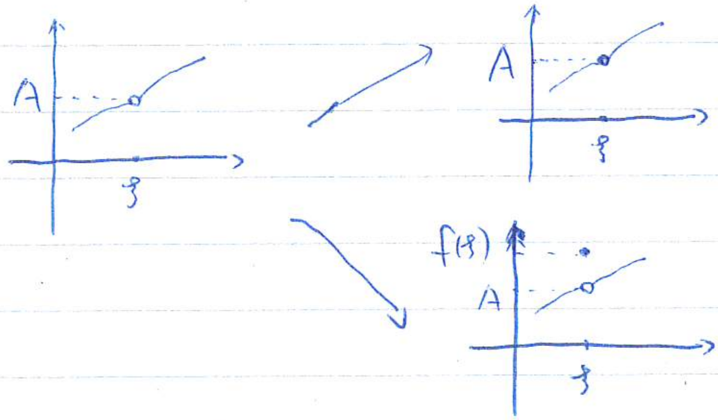
$$|f(x)-A|<\varepsilon \begin{cases} \forall x\in(\xi, \xi+\delta)\cap I \\ \forall x\in(\xi-\delta, \xi)\cap I \end{cases} \Rightarrow \forall x\in(\xi-\delta, \xi+\delta)\cap I_\xi$$

$\Rightarrow \lim_{x\to\xi} f(x) = A$. □

Now we want to put the punctured pt $f(\xi)$ back. Then ~~the in case~~ if:

$$\lim_{x\to\xi} f(x) = A \quad \& \ \xi \text{ is an interior pt},$$

we have two different cases:



$A = f(\xi)$ (I)

$A \neq f(\xi)$ (II)

We are mostly interested in case I where we have continuity of f at ξ.

Definition 2.2: Let I be an interval & $\xi \in I$. Let f be a function defined on I. We say that f is continuous at ξ if

$\lim_{x\to\xi} f(x)$ exists and equals $f(\xi)$,

i.e. $\lim_{x\to\xi} f(x) = f(\xi)$. We say f is continuous on I if it's continuous at every pts on I

Remark ①: Note f is now defined on the whole interval I, in particular $f(\xi)$ exists.

② Note by Def 2.1. $\lim_{x\to\xi} f(x) = f(\xi)$ means:

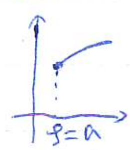
$\forall \varepsilon>0, \exists \delta>0$ s.t. $|f(x)-f(\xi)| < \cancel{\delta}\,\varepsilon \ \forall\ |x-\xi|<\delta$ & $x\in I_\xi$.

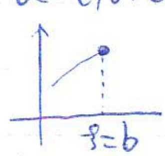
But since $|f(x)-f(\xi)|_{x=\xi} = 0 < \varepsilon \ \forall \varepsilon>0$, x can be ξ.

We may then rewrite the definition 2.2 as:

> Def 2.2: We say f is continuous at ξ if:
>
> $\forall \varepsilon>0, \exists \delta>0$, s.t. $|f(x)-f(\xi)|<\varepsilon \ \forall\ |x-\xi|<\delta$ & $x\in I$
>
> (or equivalently, $\forall\ x\in(\xi-\delta, \xi+\delta)\cap I$)

③ Note again if ξ is an end-point of a closed interval $I=[a,b]$. Then





- If $\xi=a$, $\lim_{x\to\xi} f(x) = f(\xi)$ means: $\lim_{x\to\xi+} f(x) = f(\xi+) = f(\xi)$

or $\forall \varepsilon>0, \exists \delta>0$ s.t. $|f(x)-f(\xi)|<\cancel{\delta}\,\varepsilon$, $\forall\ x\in[\xi,\xi+\delta)\cap I$

- If $\xi=b$, $\lim_{x\to\xi} f(x)=f(\xi)$ means $\lim_{x\to\xi-} f(x) = f(\xi-) = f(\xi)$

or $\forall \varepsilon>0, \exists \delta>0$ s.t. $|f(x)-f(\xi)|<\varepsilon \ \forall x\in(\xi-\delta,\xi]\cap I$.

④ Corollary (of Thm 2.1).

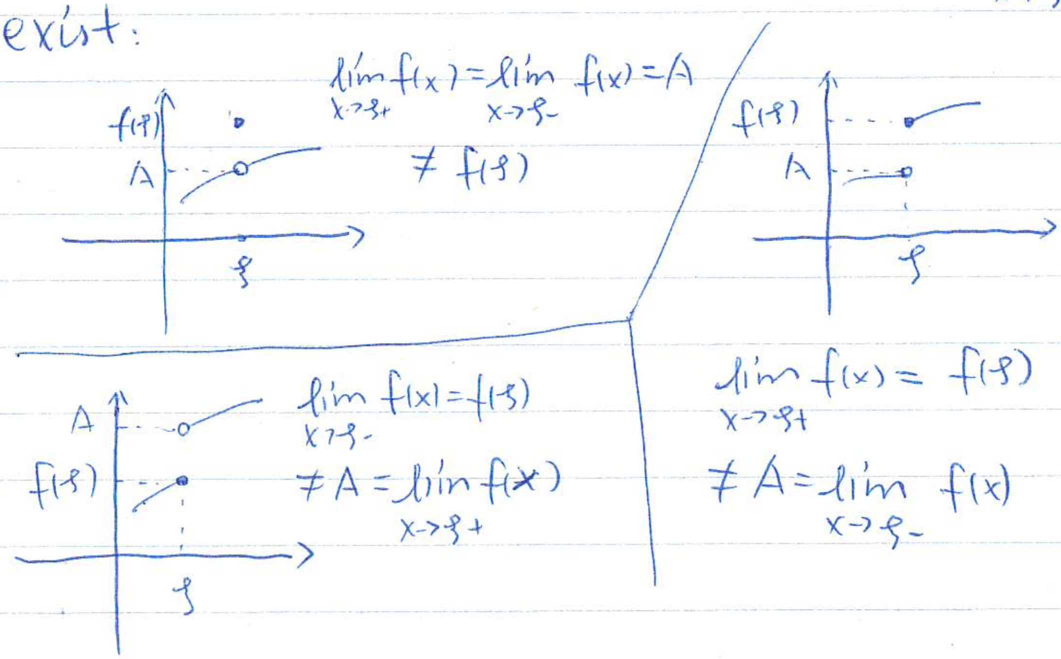
Let I be an interval & $\xi\in I$ be an interior pt. Let f be a function defined on I. Then f is continuous at ξ if and only if both left & right limits exist and equal $f(\xi)$.

~~p~~ In other words,

$$\lim_{x\to\xi} f(x) = f(\xi) \iff \lim_{x\to\xi+} f(x) = \lim_{x\to\xi-} f(x) = f(\xi)$$

Proof: In the ~~sat~~ statement of Thm 2.1, replace A by $f(\xi)$. $\square$

Case of distinnity in case $\lim_{x\to\xi-} f(x)$ & $\lim_{x\to\xi+} f(x)$ exist:



Of course, f cannot be continuous if $\lim_{x\to\xi+} f(x)$ or $\lim_{x\to\xi-} f(x)$ do not exist.

2.2.2. Examples of continuous functions

Geometrically, f is continuous at ξ means as x tends to ξ, $f(x)$ tends to $f(\xi)$. A bit more rigorously, no matter how small $\varepsilon > 0$ is, if x is sufficiently close to ξ (e.g. $\exists \delta$ s.t. $|x-\xi| < \delta$), then $f(x)$ is within ε-distance of $f(\xi)$, i.e.

$$|f(x) - f(\xi)| < \varepsilon.$$

What kind of functions are continuous? All elementary functions are continuous on the domain where they are defined.

What is an elementary function? It's a function of one variable which is the composition of a finite number of arithmetic operations $(+, -, \times, \div)$, exponentials, logarithms, and solutions of algebraic equations (e.g. $x^n = a \rightsquigarrow x = a^{\frac{1}{n}}$). There are five main types:

① $\begin{cases} f(x) = x^n & \text{power of } x \\ f(x) = x^{\frac{1}{n}} & \text{root of } x \end{cases} \rightsquigarrow f(x) = x^a,\ a \in \mathbb{R}$, <u>power functions</u>

② $f(x) = e^x$ or generally $f(x) = a^x$, $a > 0$. <u>Exponetial functions</u>

③ $f(x) = \sin(x), \cos(x), \tan(x), \ldots$ <u>Trigometric functions</u>

④ Inverse function of exponential functions

$f(x) = \ln(x)$ or generally $f(x) = \log_a(x)$

via $\begin{cases} \ln(e^x) = x \\ e^{\ln(x)} = x \end{cases}$ <u>Logarithmic functions</u>

⑤ Inverse function of trigometric functions:

$f(x) = \arcsin(x)$, or $\sin^{-1}(x)$

$\arctan(x)$, or $\tan^{-1}(x)$

All other elementary functions can be obtained from suitable combinations or compositions of these five basic types. They are all continuous on their domain. But to prove their continuity, it takes some work. In fact, we haven't defined what is x^a when $a \notin \mathbb{Q}$, $\log(x)$, ~~and~~ or $\arctan(x)$. To do all these works, we will need the things we've done in chap 1.

We start with simplest functions, i.e.

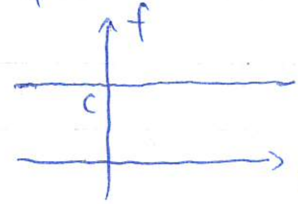
$$f(x) = x^n, \quad n \in \mathbb{N}.$$

① Example 1. First we consider the simplest case of $f(x)=x^n$, where $n=0$, i.e. $f(x)\equiv 1$. In fact, we can be a little more general:

fix $c\in\mathbb{R}$, then the constant function $f:\mathbb{R}\to\mathbb{R}$

$$f(x)\equiv c \ , \ \text{i.e. } f(x)=c \quad \forall x\in\mathbb{R}$$

is continuous on $\mathbb{R}$.



Proof: To show f is continuous on $\mathbb{R}$, we need to show f is continuous at every $\xi\in\mathbb{R}$. Fix any $\xi\in\mathbb{R}$. To show f is continuous at ξ, we need to show

$\forall \varepsilon>0, \exists \delta>0$, s.t.

$$|f(x)-f(\xi)|<\varepsilon \quad \forall x \text{ s.t. } |x-\xi|<\delta$$

Here we don't need to consider I since $I=\mathbb{R}$.

The key thing here is to find the δ. Similarly to use definition show convergence of sequence where we need to find the n_0, we work backwards:

$$|f(x)-f(\xi)|<\varepsilon \iff |c-c|<\varepsilon$$
$$\iff 0<\varepsilon$$

which is always true! Thus here δ can be any positive number. e.g. $\delta=1$. To sum up, $\forall\varepsilon>0$, let $\delta=1$, then

$$|f(x)-f(\xi)|<\varepsilon \quad \forall x \text{ s.t. } |x-\xi|<1. \qquad \square$$

Note we have alot of freedom regarding the choice of δ. But δ cannot be dependent on x. it must be a fixed constant. It can depend on ε & ξ since they are fixed

② Then we consider a slight more complicated case where $f(x) = x^n$ with $n=1$, i.e. $f(x) = x$.

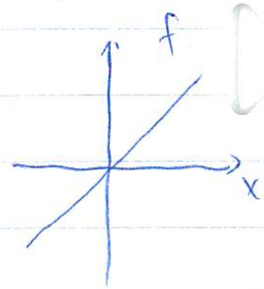
Show that $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x$ is continuous on $\mathbb{R}$

Proof: Again, we show that $\forall \xi \in \mathbb{R}$, f is continuous at ξ, i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|f(x) - f(\xi)| < \varepsilon, \quad \forall x \text{ s.t. } |x - \xi| < \delta.$$

To find the δ, we again work backwards:

$$|f(x) - f(\xi)| < \varepsilon \iff |x - \xi| < \varepsilon$$



Thus taking $\delta = \varepsilon \iff |x - \xi| < \delta$.

To sum up, $\forall \varepsilon > 0$, let $\delta = \varepsilon$, then

$$|f(x) - f(\xi)| < \varepsilon \quad \forall \ x \text{ s.t. } |x - \xi| < \delta$$

$\Rightarrow \lim_{x \to \xi} f(x) = f(\xi)$, $\forall \xi \in \mathbb{R} \Rightarrow f$ is cont. on $\mathbb{R}$ $\square$

③ Show that $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, is continuous on $\mathbb{R}$.

Proof: Fix any $\xi \in \mathbb{R}$. We want to show that $\forall \varepsilon > 0$, $\exists \delta > 0$. s.t.

$$|f(x) - f(\xi)| < \varepsilon \quad \forall \ |x - \xi| < \delta.$$

To find ~~the~~ a δ, we work backwards:

$$|f(x)-f(\xi)|<\varepsilon \iff |x^2-\xi^2|<\varepsilon$$

$$\iff |x+\xi|\cdot|x-\xi|<\varepsilon$$

Here $|x-\xi|$ can be small, but $|x+\xi|$ might not be.
To get around this issue, we first pick

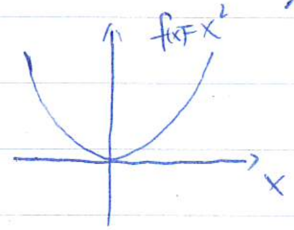
$$\delta_1=1 \text{ so that } |x-\xi|<1 \Rightarrow |x|-|\xi|<1$$

$$\Rightarrow |x|<|\xi|+1$$

$$\Rightarrow |x+\xi|\le|x|+|\xi|<|\xi|+|\xi|+1=2|\xi|+1, \text{ when } |x-\xi|<1$$

Then $\overset{\text{if } |x-\xi|<\delta_1=1}{\Longleftarrow} (2|\xi|+1)\,|x-\xi|<\varepsilon$

$$\iff |x-\xi|<\frac{\varepsilon}{2|\xi|+1}$$



Thus we take $\delta_2=\dfrac{\varepsilon}{2|\xi|+1}$. Now we can set $\delta=\min\{\delta_1,\delta_2\}$ $=\min\{1,\frac{\varepsilon}{2|\xi|+1}\}$

We have $\forall\varepsilon>0$, if $\delta=\min\{1,\frac{\varepsilon}{2|\xi|+1}\}$, then

$$|f(x)-f(\xi)|<\varepsilon \quad \forall\ |x-\xi|<\delta$$

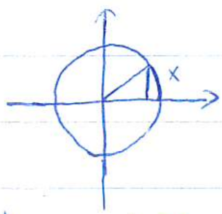
$\Rightarrow \lim\limits_{x\to\xi} f(x)=f(\xi)\ \ \forall\xi\in\mathbb{R} \Rightarrow f$ is continuous on $\mathbb{R}^2$.

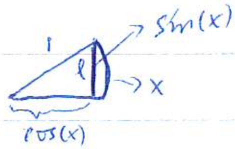
□

In general, we can use similar proof to show continuity of $f(x)=x^n$, $\forall n\in\mathbb{N}$. But we'd like to prove it in section 2.2 as it will be much easier.

Example 2. Show that $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$, is continuous on $\mathbb{R}$.

Proof: What is $\sin(x)$? When the arc length is x for $0 \le x \le \frac{\pi}{2}$, $\sin(x)$ is the length of line segament ℓ.



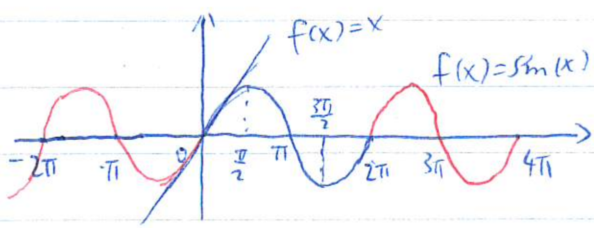


Then $\forall\ \frac{\pi}{2} \le x \le \pi$, $\sin(x) = \sin(\pi - x)$ $\qquad \frac{\pi}{2} \le x \le \pi \Rightarrow 0 \le \pi - x \le \frac{\pi}{2}$

$\forall\ \pi \le x \le \cancel{\tfrac{3\pi}{2}}\ 2\pi$, $\sin(x) = -\sin(2\pi - x)$ $\qquad \pi \le x \le 2\pi \Rightarrow 0 \le 2\pi - x \le \pi$

Thus $\sin(x)$ is defined on $0 \le x \le 2\pi$.

Then we extend the definition of $\sin(x)$ to $\mathbb{R}$ via ~~f()~~ $\sin(x + 2\pi n) = \sin(x)$



Then we may define $\cos(x) = \sin(\frac{\pi}{2} - x)$

In particular, we note the follow facts we've learned about $\sin(x)$.

① $\sin a - \sin b = 2 \sin \frac{a-b}{2} \cos \frac{a+b}{2}$

② $|\sin a| \le |a|$, $\forall\ a \in \mathbb{R}$

③ $|\sin a| \le 1$, $\forall\ a \in \mathbb{R}$

$\Rightarrow |\cos(a)| \le 1$, $\forall\ a \in \mathbb{R}$

Now fix any $\xi \in \mathbb{R}$, we want to show $f(x) = \sin(x)$ is continuous at ξ, i.e.

$\forall \varepsilon > 0$, $\exists \delta > 0$. s.t.

$$|\sin(x) - \sin(\xi)| < \varepsilon, \ \forall |x - \xi| < \delta.$$

To find δ, we work backwards:

$$|\sin(x) - \sin(\xi)| < \varepsilon \overset{①}{\Longleftrightarrow} \left|2\sin\frac{x-\xi}{2}\cos\frac{x+\xi}{2}\right| < \varepsilon$$

$$\Longleftrightarrow 2\left|\cos\frac{x+\xi}{2}\right| \cdot \left|\sin\frac{x-\xi}{2}\right| < \varepsilon$$

$$\overset{③}{\Longleftarrow} 2\left|\sin\frac{x-\xi}{2}\right| < \varepsilon$$

$$\overset{②}{\Longleftarrow} 2\left|\frac{x-\xi}{2}\right| < \varepsilon$$

$$\Longleftrightarrow |x - \xi| < \varepsilon$$

Thus δ can be chosen as ε. To sum up.

$\forall \varepsilon > 0$, if we set $\delta = \varepsilon$, then

$$|f(x) - f(\xi)| < \varepsilon \quad \forall |x - \xi| < \delta$$

$\Rightarrow \lim_{x \to \xi} f(x) = f(\xi) \quad \forall \xi \in \mathbb{R} \Rightarrow f$ is continuous on $\mathbb{R}$ $\square$

Similarly, we can show $f(x) = \cos(x)$ is continuous on $\mathbb{R}$. The only difference is

$$\cos(a) - \cos(b) = 2\sin\frac{a+b}{2}\sin\frac{a-b}{2}$$

2.2.3. Relation between convergent sequences & continuous functions

First of all, we want to explore a little bit what means by saying $f(x)$ does not tend to $f(\xi)$ as x tends to ξ.

In general, the ~~tg~~ logical opposite of $\forall$ is $\exists$

- - - - - - - - - of $\exists$ is $\forall$.

Note $\lim\limits_{x \to \xi} f(x) = f(\xi)$ means:

$\forall \varepsilon > 0, \exists \delta > 0$, s.t. $|f(x) - f(\xi)| < \varepsilon$, $\forall\ |x - \xi| < \delta$.

Thus $f(x) \not\to f(\xi)$ as $x \to \xi$ means:

$\exists \varepsilon_0 > 0$, $\forall \delta > 0$, s.t. $|f(x_0) - f(\xi)| \geq \varepsilon_0$, $\exists\ |x_0 - \xi| < \delta$

Roughly speaking, if there is a $\varepsilon_0 > 0$, such that no matter how small δ is, one can find a $x_0 \in (\xi - \delta, \xi + \delta)$ with $|f(x_0) - f(\xi)| \geq \varepsilon_0$, then $f(x) \not\to f(\xi)$.

In fact, $\lim\limits_{n \to \infty} a_n = L$ means $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{Z}_+$ s.t.

$|a_n - L| < \varepsilon$, $\forall n \geq n_0$

Thus $a_n \not\to L$ as $n \to \infty$ means: $\exists \varepsilon_0 > 0$, $\forall n_0 \in \mathbb{Z}$ s.t.

$|a_n - L| \geq \varepsilon_0$, $\exists n \geq n_0$

Theorem 2.2. Let $f: I \to \mathbb{R}$ be ~~cont~~ a function. Let $\xi \in I$. Then f is continuous at ξ if & only if the following statement is true:

$$(*) \left\{ \begin{array}{l} \forall \{x_n\}_{n\geq 1} \text{ on } I \text{ with } \lim_{n\to\infty} x_n = \xi, \ \{f(x_n)\}_{n\geq 1} \text{ is} \\ \text{convergent } \& \lim_{n\to\infty} f(x_n) = f(\xi) \end{array} \right\}$$

Proof: ① "Only if part", i.e.

$\lim_{x\to\xi} f(x) = f(\xi) \Rightarrow (*)$ statement.

By definition of $\lim_{x\to\xi} f(x) = f(\xi)$, $\forall \varepsilon > 0$, $\exists \delta > 0$

s.t. $|f(x) - f(\xi)| < \varepsilon$, $\forall |x - \xi| < \delta$, $x \in I$.

Now let $\{x_n\}_{n\geq 1}$ be a sequence in I s.t. $\lim_{n\to\infty} x_n = \xi$

Then for the δ above, $\exists n_0$ s.t.

$$|x_n - \xi| < \delta, \ \forall n \geq n_0 \quad (\text{of course } x_n \in I)$$

Thus by the choice of δ, we have

$$|f(x_n) - f(\xi)| < \varepsilon, \ \forall n \geq n_0$$

To sum up, $\forall \varepsilon > 0$, $\exists n_0$ s.t.

$$|f(x_n) - f(\xi)| < \varepsilon, \ \forall n \geq n_0$$

$\Rightarrow \lim_{n\to\infty} f(x_n) = f(\xi)$

② "If part", i.e.

$(*) \Rightarrow \lim_{x \to \xi} f(x) = f(\xi)$.

Assume for the sake of arguing by contradiction, $f(x) \not\to f(\xi)$ as $x \to \xi$. Then $\exists\, \varepsilon_0 > 0$, s.t. $\forall\, \delta > 0$, $\exists\, x_0$ s.t. $|x_0 - \xi| < \delta$ & $|f(x_0) - f(\xi)| \geq \varepsilon_0$

In particular, $\forall\, \frac{1}{n}, n \in \mathbb{Z}_+$, $\exists\, x_n$ s.t. $|x_n - \xi| < \frac{1}{n}$

& $|f(x_n) - f(\xi)| \geq \varepsilon_0$

Thus $\lim_{n \to \infty} x_n = \xi$ since $-\frac{1}{n} \leq x_n - \xi \leq \frac{1}{n}$, $\forall n \geq \mathbb{Z}_+$

And $|f(x_n) - f(\xi)| \geq \varepsilon_0$ $\forall n \geq 1$. $\Rightarrow f(x_n) \not\to f(\xi)$ as $n \to \infty$.

Thus we've found a sequence $\{x_n\}_{n \geq 1}$ in I s.t.

$\lim_{n \to \infty} x_n = \xi$ & $f(x_n) \not\to f(\xi)$ as $n \to \infty$,

contradicts with $(*)$. Thus our assumption $f(x)$ is not continuous at ξ is false

$\Rightarrow f$ is continuous at ξ $\square$

Remark: ① Again what is continuity (at ξ)?

It means as x approaches ξ, $f(x)$ approaches $f(\xi)$. One way to say "approching" is to use convergent sequences. Thus Thm 2.2 makes perfect sense: f is continuous at ξ $\Leftrightarrow$ $\forall \{x_n\}$ s.t. $x_n \to \xi$ as $n \to \infty$, $f(x_n) \to f(\xi)$ as $n \to \infty$.

In summary, a continuous function sends convergent sequences to convergent sequences. Moreover, the limit of the image sequence is the image of the limit of the original sequence, i.e.

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(\xi)$$

We shall see that Thm 2.2 is extremely powerful. Because it reduces "continuity" to "~~convergent~~ convergence of sequences." As a consequence, we can use all the results of Chap 1 to help ~~us~~ us determine continuity.

② Similarly to the statement & proof of Thm 2.2, we have:

$\lim_{x\to\xi} f(x) = A \Leftrightarrow \forall \{x_n\}_{n\geq 1}, \lim_{n\to\infty} x_n = \xi$, and $x_n \neq \xi$, it holds

$$\lim_{n\to\infty} f(x_n) = A.$$

Similarly for $\lim_{x\to\xi+} f(x)$ or $\lim_{x\to\xi-} f(x)$.

We use one example to show the power of Thm 2.2.

Example 3. Consider $f(x) = \sin(\frac{1}{x}) : (0, +\infty) \to \mathbb{R}$

Show that $\lim_{x \to 0+} f(x)$ does not exist.

Proof: Suppose there is a $A \in \mathbb{R}$ s.t.

$$\lim_{x \to 0+} f(x) = A.$$

Then by Thm 2.2, $\forall \{x_n\}_{n \geq 1}$, $\lim_{n \to \infty} x_n = 0$, $x_n > 0$

it holds that $\lim_{n \to \infty} f(x_n) = A$.

Now we first take $x_n^{(1)} = \frac{1}{n\pi}$, $n \geq 1$. Then

$\lim_{n \to \infty} x_n^{(1)} = \lim_{n \to \infty} \frac{1}{n\pi} = \frac{1}{\pi} \lim_{n \to \infty} \frac{1}{n} = 0$ & $x_n > 0$ $\forall n \geq 1$

And $f(x_n^{(1)}) = \sin\left(\frac{1}{1/n\pi}\right) = \sin(n\pi) = 0$, $\forall n \geq 1$

$\Rightarrow \lim_{n \to \infty} f(x_n) = 0$

Take $x_n^{(2)} = \frac{1}{2n\pi + \frac{\pi}{2}}$ $n \geq 1$. Then $\lim x_n = \lim \frac{1}{2n\pi + \frac{\pi}{2}} = 0$

& $x_n > 0$ $\forall n \geq 1$.

And $f(x_n^{(2)}) = \sin\left(\frac{1}{\frac{1}{2n\pi + \frac{\pi}{2}}}\right) = \sin(2n\pi + \frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$ $\forall n \geq 1$

$\Rightarrow \lim_{n \to \infty} f(x_n^{(2)}) = 1$

Since we have two different limits 0 & 1 ~~from~~ for two sequences $\{x_n^{(1)}\}$ & $\{x_n^{(2)}\}$, $\lim_{x \to 0+} f(x)$ doesn't exist. $\square$

2.2 Operations with Continuous Functions

Let $f, g : I \to \mathbb{R}$ be two functions defined on a common interval I. Then functions
$f \pm g$, $f \cdot g$, and f/g (if $g \neq 0$) $: I \to \mathbb{R}$
are defined by

$$(f \pm g)(x) = f(x) \pm g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x) \qquad \forall x \in \mathbb{R}$$
$$(f/g)(x) = f(x)/g(x) \quad (g(x) \neq 0)$$

Theorem 2.3 (Sum of continuous functions are continuous). Let $f, g : I \to \mathbb{R}$ be continuous at $\xi \in I$. Then $f+g$ is continuous at ξ. In particular, if f & g are continuous on I, then $f+g$ is continuous on I.

Proof: Here we are going to have two different proofs. The first is proof by definition. The second is proof via Theorem 2.2.

First proof: Since f and g are continuous at ξ, we have by definition $\forall \varepsilon > 0$.

$\exists \delta_1 > 0$ s.t. $|f(x) - f(\xi)| < \frac{\varepsilon}{2}$ $\forall |x - \xi| < \delta_1$ and $x \in I$;

$\exists \delta_2 > 0$ s.t. $|g(x) - g(\xi)| < \frac{\varepsilon}{2}$ $\forall |x - \xi| < \delta_2$ and $x \in I$.

Set $\delta = \min\{\delta_1, \delta_2\}$. Then if $|x - \xi| < \delta$ and $x \in I$, we have both $|f(x) - f(\xi)| < \varepsilon/2$ and $|g(x) - g(\xi)| < \varepsilon/2$.

Then $\forall\ |x-\xi|<\delta$ and $x\in I$, we have

$$|(f+g)(x)-(f+g)(\xi)| = |f(x)+g(x)-(f(\xi)+g(\xi))|$$
$$= |f(x)-f(\xi)+g(x)-g(\xi)|$$
$$\leq |f(x)-f(\xi)|+|g(x)-g(\xi)|$$
$$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}$$
$$= \varepsilon$$

To sum up. $\forall \varepsilon>0$, we've found a $\delta>0$ s.t.
$|(f+g)(x)-(f+g)(\xi)|<\varepsilon \quad \forall\ |x-\xi|<\delta$ and $x\in I$.

$\Rightarrow \lim\limits_{x\to\xi}(f+g)(x) = (f+g)(\xi)$, i.e. $f+g$ is continuous at ξ.

Second proof: By Theorem 2.2, "if $\forall \{x_n\}_{n\geq 1}$ s.t. $x_n\in I\ \forall n\geq 1$ & $\lim\limits_{n\to\infty} x_n=\xi$, we can show

$$\lim_{n\to\infty}(f+g)(x_n) = (f+g)(\xi).$$

then $(f+g)$ is continuous at ξ".

To this end, we fix any $\{x_n\}_{n\geq 1}$ s.t. $x_n\in I\ \forall n\geq 1$ & $\lim\limits_{n\to\infty} x_n=\xi$.

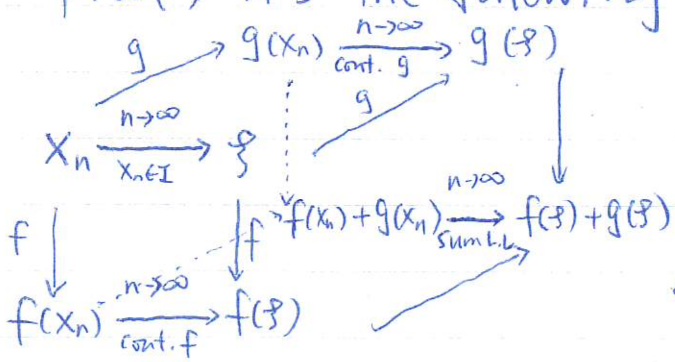
By Theorem 2.2 & f, g are continuous at ξ, we have $\lim\limits_{n\to\infty} f(x_n) = f(\xi)$

$\lim\limits_{n\to\infty} g(x_n) = g(\xi)$

By sum limit law, we then have

$$\lim_{n\to\infty}(f+g)(x_n) = \lim_{n\to\infty}\big(f(x_n)+g(x_n)\big)$$
$$= \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n)$$
$$= f(\xi) + g(\xi)$$
$$= (f+g)(\xi), \text{ done!} \qquad \square$$

If we want to use a diagram to illustrate the second proof, it's the following:



$\Rightarrow \lim\limits_{n\to\infty}(f+g)(x_n) = (f+g)(\xi)$

$\Leftrightarrow f+g$ is cont. at ξ.

Theorem 2.4 (Product of continuous functions are continuous)
$f, g: I \to \mathbb{R}$ are cont. at $\xi \in I \Rightarrow f\cdot g$ is cont. at ξ.

Proof: Here we are going to give the proof by definition and leave the second proof as a homework problem.

By continuity of f & g at ξ, we have:

$\forall \varepsilon > 0$, $\exists \delta_1 > 0$ s.t. $|f(x) - f(\xi)| < \boxed{\frac{1}{2|f(\xi)|+2}}\varepsilon \quad \forall |x-\xi| < \delta_1, x \in I$.

$\exists \delta_2 > 0$ s.t. $|g(x) - g(\xi)| < \boxed{\frac{1}{2|g(\xi)|+1}}\varepsilon \quad \forall |x-\xi| < \delta_2, x \in I$

Here similar to the proof of product limit law, we need a little more fact:

Again by continuity of f at ξ, we can find $\delta_0 > 0$ s.t. $|f(x) - f(\xi)| < 1 \quad \forall |x - \xi| < \delta_0, x \in I$.

By triangle inequality, $|f(x)| - |f(\xi)| \le |f(x) - f(\xi)| < 1$

$$\Rightarrow |f(x)| \le |f(\xi)| + 1, \quad \forall |x-\xi| < \delta_0, x \in I.$$

Now we set $\delta = \min\{\delta_0, \delta_1, \delta_2\}$, then

$\forall |x - \xi| < \delta$ & $x \in I$, we have

$$
\begin{aligned}
|(f\cdot g)(x) - (f\cdot g)(\xi)| &= |f(x)\cdot g(x) - f(\xi)\cdot g(\xi)| \\
&= |f(x)\cdot g(x) - f(x)\cdot g(\xi) + f(x)\cdot g(\xi) - f(\xi)\cdot g(\xi)| \\
&\le |f(x)\cdot g(x) - f(x)\cdot g(\xi)| + |f(x)\cdot g(\xi) - f(\xi)\cdot g(\xi)| \\
&= |f(x)|\cdot|g(x) - g(\xi)| + |g(\xi)|\cdot|f(x) - f(\xi)| \\
&\le (|f(\xi)|+1)\frac{\varepsilon}{2(|f(\xi)|+1)} + |g(\xi)|\frac{\varepsilon}{2|g(\xi)|+1} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$, s.t. $|(f\cdot g)(x) - (f\cdot g)(\xi)| < \varepsilon \quad \forall |x - \xi| < \delta, x \in I$ $\square$

Theorem 2.5 (Quotient of continuous functions are continuous)

$f, g: I \to \mathbb{R}$ cont. at ξ and $g(\xi) \neq 0 \Rightarrow \frac{f}{g}$ is cont. at ξ.

Proof: Here we use the second proof (by Thm 2.2) and leave the proof by definition as a homework problem.

By Thm 2.2, $\frac{f}{g}$ is cont. at ξ if:

"$\forall \{x_n\}$ s.t. $x_n \in I$ $\forall n \geq 1$ & $\lim_{n\to\infty} x_n = \xi$, we have $\lim_{n\to\infty} \left(\frac{f}{g}\right)(x_n) = \left(\frac{f}{g}\right)(\xi)$." Fix such a $\{x_n\}_{n\geq 1}$.

Since f & g are cont. at ξ & $g(\xi) \neq 0$, by Thm 2.2

we have $\lim_{n\to\infty} f(x_n) = f(\xi)$

$\lim_{n\to\infty} g(x_n) = g(\xi)$

By quotient limit law, we have

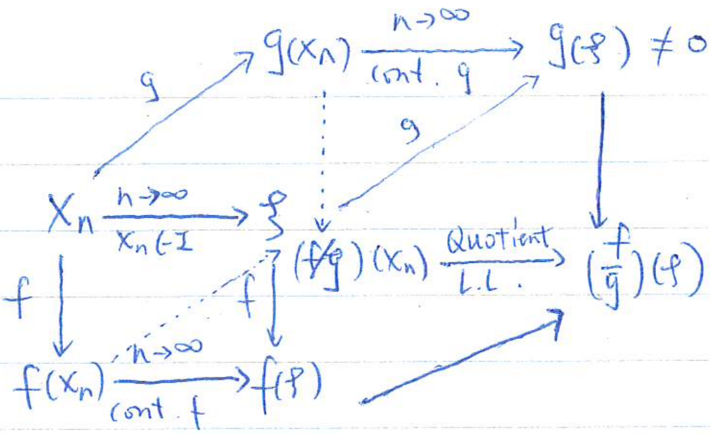
$$\lim_{n\to\infty} \left(\frac{f}{g}\right)(x_n) = \lim_{n\to\infty} \frac{f(x_n)}{g(x_n)}$$

$$= \frac{\lim f(x_n)}{\lim g(x_n)}$$

$$= \frac{f(\xi)}{g(\xi)}$$

$$= \left(\frac{f}{g}\right)(\xi).$$ done! $\square$

use a diagram:



$\forall \{x_n\}$, $\lim x_n = \xi$, $x_n \in I, \forall n \geq 1$,

$\Rightarrow \lim_{n\to\infty} \left(\frac{f}{g}\right)(x_n) = \lim_{n\to\infty} \left(\frac{f}{g}\right)(\xi)$

Definition 2.3 (composition of functions)

Let $g: I \to \mathbb{R}$ be a function. Let $J_0 = \{g(x) : x \in I\}$, i.e. the range of g. Let J be an interval s.t. $J_0 \subset J$. Let $f: J \to \mathbb{R}$ be another function. Then for each $x \in I$, there corresponds a number

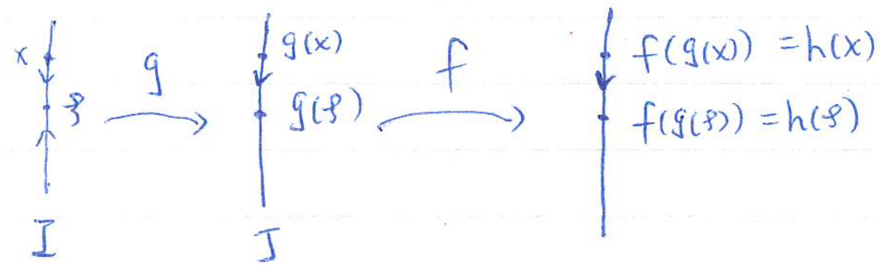
$$f(g(x)) \in \mathbb{R}.$$

Thus we can define a new function $h(x) := f(g(x))$ on I. It's called the composition of f with g, denoted by $h = f \circ g$.

Theorem 2.6 (composition of continuous functions are continuous)

Let f and g be as in def 2.3. Suppose g is continuous at $\xi \in I$ & f is continuous at $g(\xi) \in J$. Then $h(x) = f \circ g(x)$ is continuous at $\xi \in I$.

Proof: Again we may have two different proofs. Geometrically,



First proof by definition:

f is cont. at $g(\xi) \Rightarrow \forall \varepsilon > 0, \exists \eta > 0$ s.t.

$$|f(y) - f(g(\xi))| < \varepsilon \quad \forall\ |y - g(\xi)| < \eta \ \&\ y \in J$$

g is cont. at $\xi \Rightarrow$ For the $\eta > 0$ above, $\exists \delta > 0$ s.t.

$$|g(x) - g(\xi)| < \eta, \quad \forall\ |x - \xi| < \delta \ \&\ x \in I.$$

Clearly $g(x) \in J$ $\forall x \in I$ by definition.

Thus $\forall \varepsilon > 0$, $|f(g(x)) - f(g(\xi))| < \varepsilon \quad \forall\ |x - \xi| < \delta \ \&\ x \in I$

$\Rightarrow \underset{h(x)=}{f(g(x))}$ is continuous at ξ.

Second proof: Fix any $\{x_n\}_{n\geq 1}$ s.t. $x_n \in I$ $\forall n \geq 1$ & $\lim_{n\to\infty} x_n = \xi$

By continuity of g & Thm 2.2 $\Rightarrow \lim_{n\to\infty} f(x_n) = g(\xi)$

$g(x_n) \in J$, $\forall n \geq 1$

By continuity of f & Thm 2.2

$\Rightarrow \lim_{n\to\infty} f(g(x_n)) = \lim_{n\to\infty} f(g(\xi)) \Rightarrow f\circ g$ is cont. at ξ by Thm 2.2 $\square$

$$\begin{tikzcd} x_n \arrow[r, "n\to\infty"] \arrow[d, "g"'] \arrow[dd, bend right] & \xi \arrow[d] \arrow[dd, bend left] \\ g(x_n) \arrow[r, "n\to\infty", "\text{cont. } g"'] \arrow[d, "f"'] & g(\xi) \arrow[d, "f"] \\ f(g(x_n)) \arrow[r, "n\to\infty"', "\text{cont. } f"'] & f(g(\xi)) \end{tikzcd} \Rightarrow \begin{tikzcd} x_n \arrow[r, "n\to\infty"] \arrow[d, "f\circ g"'] & \xi \arrow[d, "f\circ g"] \\ f\circ g(x_n) \arrow[r, "n\to\infty"] & f\circ g(\xi) \end{tikzcd} \Rightarrow f\circ g \text{ is cont at } \xi.$$

Application of all the theorems:

First we note that we can extend the sum & product of two functions to the sum & product of any finite number of functions.

Example 1: Recall we proved by definition that $f(x) = c, x, x^2$ are continuous.

① $f(x) = x^n$, $n \geq 0$ is continuous on $\mathbb{R}$.

Proof: $f(x) = \underbrace{x \cdot x \cdots x}_{n}$ is a product of n functions $g(x) = x$ which is continuous $\Rightarrow f(x)$ is continuous.

② Consider $p(x) = \sum_{k=0}^{n} a_k \cdot x^k$ a polynomial of degree n. Here $a_0, \dots, a_n$ are ~~contr~~ constants. $p(x)$ is continuous on $\mathbb{R}$.

Proof: For each k, $a_k \cdot x^k$ is continuous since it's a product of $f(x) \equiv a_k$ & $g(x) = x^k$ which are continuous. Thus

$$p(x) = a_0 + a_1 \cdot x + \cdots + a_n x^n$$

is continuous on $\mathbb{R}$ since it's a finite sum of continuous functions.

③ Even more general, we call $r(x) = \frac{p(x)}{q(x)}$ a rational function if both p & q are polynomials. Note r is not defined on such x's where $q(x) = 0$. But $r(x)$ is continuous at every point where it's well-defined.

Proof: At every point x where r is well defined, it's a quotient of two continuous functions with nonzero denominator. Thus it's continuous at every such x.

Example 2: $\tan(x) = \frac{\sin(x)}{\cos(x)}$ is continuous at every $x \neq n\pi + \frac{\pi}{2}$. $f(x) = \sin(x^2)$ is continuous on $\mathbb{R}$. Similarly $f(x) = \sin(\frac{1}{x})$ is continuous at every $x \neq 0$. $f(x) = \sin(\sqrt{x}) : [0, +\infty) \to \mathbb{R}$ is cont.

Proof: we just need to apply all the Theorems 2.3 - 2.6. □

Example 3: Compute $\lim\limits_{n \to \infty} \sin(\frac{1}{\sqrt{n+1}})$.

Solution. Note $\sin(\frac{1}{\sqrt{n+1}}) = \sin\sqrt{\frac{1}{n+1}}$.

We know from example 2, $f(x) = \sin\sqrt{x} : [0, +\infty) \to \mathbb{R}$ is continuous on $[0, +\infty)$.

Thus by Theorem 2.2, $\forall \{x_n\}_{n \geq 1}$ $x_n \geq 0$ $\lim\limits_{n \to \infty} x_n = 0$ we have $\lim\limits_{n \to \infty} f(x_n) = f(0)$.

Now we know $\lim\limits_{n \to \infty} \frac{1}{n+1} = 0$ & $\frac{1}{n+1} \geq 0$ $\forall n \geq 1$.

$\Rightarrow \lim\limits_{n \to \infty} \sin(\frac{1}{\sqrt{n+1}}) = \lim\limits_{n \to \infty} f(\frac{1}{n+1}) = f(0) = \sin\sqrt{0} = 0$

□

2.3 Maximum & Minimum

Definition 2.4. Let $f: I \to \mathbb{R}$ be a function defined on an interval $I \subseteq \mathbb{R}$.

(1) The range of f is $\{f(x) : x \in I\}$, denoted by $f(I)$, which is a subset of $\mathbb{R}$.

(2) If $f(I)$ is bounded above, then we say f is bounded above. By definition, it means $\exists M \in \mathbb{R}$ s.t. $f(x) \leq M \quad \forall x \in I$.

Then by the least-upper-bound property of $\mathbb{R}$ we introduced in Section 1.4, if f is bounded above, $f(I)$ has a least upper bound, or supremum, denoted by $\sup_{x \in I} f(x)$. By definition

$\alpha = \sup_{x \in I} f(x)$ if
- $f(x) \leq \alpha \ \forall x \in I$ (α is an upper bound of $f(I)$).
- $\forall \varepsilon > 0, \exists x \in I$ s.t. $f(x) > \alpha - \varepsilon$ (No number smaller than α can be an upper bound of $f(I)$).

(3). If $f(I)$ is bounded below, then we say f is bounded below, i.e. $\exists m \in \mathbb{R}$ s.t. $f(x) \geq m \quad \forall x \in I$.
Similar to (2), if f is bounde below, then $f(I)$ has ~~a~~ the greatest lower bound, or infimum.

denoted by $\inf_{x\in I} f(x)$. By definition, $\beta = \inf_{x\in I} f(x)$ if

- $f(x) \geq \beta \ \forall x \in I$ (β is a lower bound of $f(I)$)
- $\forall \varepsilon > 0, \exists x \in I$ s.t. $f(x) < \beta + \varepsilon$ (No number bigger than β can be a lower bound of $f(I)$).

A natural question is: what kind of functions are bounded?

(4) If $f(I)$ is bounded, i.e. bounded both above & below, then we say f is bounded. Note f is bounded if $\exists M > 0$ s.t. $|f(x)| \leq M \ \forall x \in I$ □

It's related to both f & I. The following Theorem provides us with a large class of functions that are bounded.

Theorem 2.7: Let f be a continuous function defined on a closed, bounded interval $I = [a, b]$. Then f is bounded.

Proof: Argue by contradiction. Suppose f is not bounded. Note f is bounded if $\exists$ $M > 0$ s.t. $|f(x)| \leq M \ \forall x \in I$. Thus

f is not bounded means.

$\forall M>0, \exists x \in I$ s.t. $|f(x)| > M$

In particular, $\forall n \in \mathbb{Z}_+, \exists x_n \in I$ s.t. $|f(x_n)| > n$.

$\{x_n\}_{n\geq 1}$ is sequence in $I \Rightarrow \{x_n\}_{n\geq 1}$ is bounded.

By Bolzano-Weierstrass Thm (for sequences).

$\{x_n\}_{n\geq 1}$ has a convergent subsequence

$$\{x_{n_k}\}_{k\geq 1}$$

Thus $\exists \xi \in \mathbb{R}$ s.t. $\lim_{k\to\infty} x_{n_k} = \xi$.

Claim: $\xi \in I$.

Indeed, if $\exists k_0$ s.t. $x_{n_{k_0}} = \xi$, then $\xi = x_{n_{k_0}} \in I$.

if $x_{n_k} \neq \xi$ $\forall k\geq 1$, then by definition

$\forall \varepsilon>0, \exists K_0 \in \mathbb{Z}_+$ s.t. $|x_{n_k} - \xi| < \varepsilon$, $\forall k \geq K_0$

Since $x_{n_k} \in I$ & $x_{n_k} \neq \xi$, we have ξ is a limit point of I. Since I is closed, we obtain $\xi \in I' \subset I$.

Now by continuity & Theorem 2.2.

$$\lim_{k\to\infty} f(x_{n_k}) = f(\xi)$$

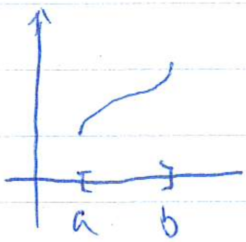
In particular, $\{f(x_{n_k})\}_{k\geq 1}$ is bounded. But

~~$|f(n_k)| > n_k \geq k, \forall k \in \mathbb{Z}_+$ and $\{n_k\}$ is a~~
~~where $\Rightarrow \{f(n_k)\}$~~
~~$|f(n_k)| > n_1$.~~

$$|f(x_{n_k})| > n_k \geq k \quad \forall k \geq 1.$$

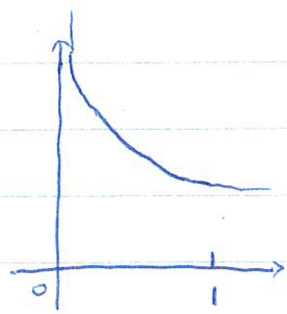
$\Rightarrow \{f(x_{n_k})\}_{k\geq 1}$ is unbounded ~~which cannot be convergent~~ contradiction $\square$

Geometrically



Theorem 2.7 is not true if I is not closed

Example 1: Let $f(x) = \frac{1}{x} : (0,1] \to \mathbb{R}$ is not bounded.



Proof: Taking $\{\frac{1}{n}\}_{n\geq 1}$ which is a sequence in $(0,1]$. Then

$$f(\tfrac{1}{n}) = \frac{1}{\frac{1}{n}} = n \text{ is unbounded}$$

$\Rightarrow f$ is unbounded.

Definition 2.5: Let $f: I \to \mathbb{R}$ be bounded above. Suppose $\exists\, \xi \in I$ s.t. $f(x) \leq f(\xi)$ $\forall x \in I$. Then we call ξ a maximum point of f and we call $f(\xi)$ the maximum of f on I. We also that f has a maximum, and that it assumes its maximum on I at ξ. Similarly, we say ξ is a minimum point of f and $f(\xi)$ a minimum of f if $f(x) \geq f(\xi)$ $\forall x \in I$.

Remark: ① If $f(\xi)$ is the maximum of f on I, then it's also the supremum of f on I.

Indeed (i) $f(x) \leq f(\xi)$, $\forall x \in I$

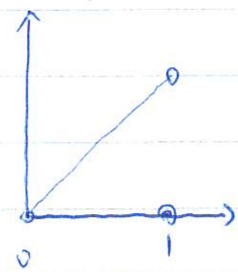
(ii). $\forall \varepsilon > 0$, $f(\xi) > f(\xi) - \varepsilon$ $\Rightarrow f(\xi) = \sup\limits_{x \in I} f(\xi)$.

Similarly, if $f(\xi)$ is the minimum of f on I, then $f(\xi) = \inf\limits_{x \in I} f(I)$.

② On the other, if f is bounded on I, then f may not have minimum or maximum on I.

Example 2. $f(x) = x : (0,1) \to \mathbb{R}$.
does not have minimum or maximum on $(0,1)$.

Proof: Clearly, $0 = \inf_{x \in I} f(x)$ & $1 = \sup_{x \in I} f(x)$. However, $\forall x \in (0,1)$ $0 < f(x) < 1$. Thus f has no minimum or maximum on $(0,1)$. Because otherwise it would assume its infimum or supremum at some $\xi \in (0,1)$.



A natural question: what kind of functions have maximum and minimum? Of course they must be bounded first. It's again related to both f & I.

Theorem 2.8. Let f be a continuous function defined on a closed, bound interval $I = [a,b]$. Then f has minimum & maximum on I.

Proof: By Thm 2.7, f is bounded on I, i.e. $\exists M > 0$ s.t. $|f(x)| < M$ $\forall x \in I$.

In particular, f is bound above & has its supremum on I. Let $\alpha = \sup\limits_{x \in I} f(x)$.

We will show α is the maximum of f, i.e. $\exists\, \xi \in I$ s.t. $f(\xi) = \alpha$.

By definition of supremum, $\forall\, \varepsilon > 0$, $\exists\, x_\varepsilon \in I$ s.t. $\alpha - \varepsilon < f(x_\varepsilon) \leq \alpha$. In particular, taking $\varepsilon = \frac{1}{n}$, we find for each $n \in \mathbb{Z}_+$ a x_n s.t.

$$\alpha - \frac{1}{n} < f(x_n) < \alpha$$

$\Rightarrow \lim\limits_{n \to \infty} f(x_n) = \alpha$ by squeeze Thm. $\{x_n\}$ is bounded.

$\Rightarrow$ By Bolzano-Weierstrass, $\exists$ a subsequence $\{x_{n_k}\}_{k \geq 1}$ of $\{x_n\}_{n \geq 1}$ that is convergent, i.e. $\exists\, \xi \in \mathbb{R}$ s.t. $\lim\limits_{k \to \infty} x_{n_k} = \xi$. Then by the proof of Thm 2.7, we know $\xi \in I$. By continuity & Thm 2.7, we now have

$$f(\xi) = \lim_{k \to \infty} f(x_{n_k}) = \lim_{n \to \infty} f(x_n) = \alpha$$

$\{f(x_{n_k})\}$ is subsequence of $\{f(x_n)\}$.

as desired. The proof of f has minimum on I is completely similar. One only need to use that if $\beta = \inf_{x \in I} f(x)$, then $\forall \varepsilon > 0$, $\exists\, x_\varepsilon \in I$ s.t. $\beta \le f(x_\varepsilon) < \beta + \varepsilon$.

Then one can take $\varepsilon = \frac{1}{n}$ & obtain a sequence $x_n \in I$ s.t. $\beta \le f(x_n) < \beta + \frac{1}{n}$

$\Rightarrow f(x_n) \to \beta$ as $n \to \infty$.

Again by Bolzano Weierstrass & boundedness of $\{x_n\}$, we can find $\{x_{n_k}\}$ that is convergent &

$$\lim_{k \to \infty} x_{n_k} = \xi \in I.$$

Thus $f(\xi) = \lim_{k \to \infty} f(x_{n_k}) = \lim_{k \to \infty} f(x_n) = \beta$

$\Rightarrow \beta$ is the ~~minium~~ minimum of f on I. $\square$

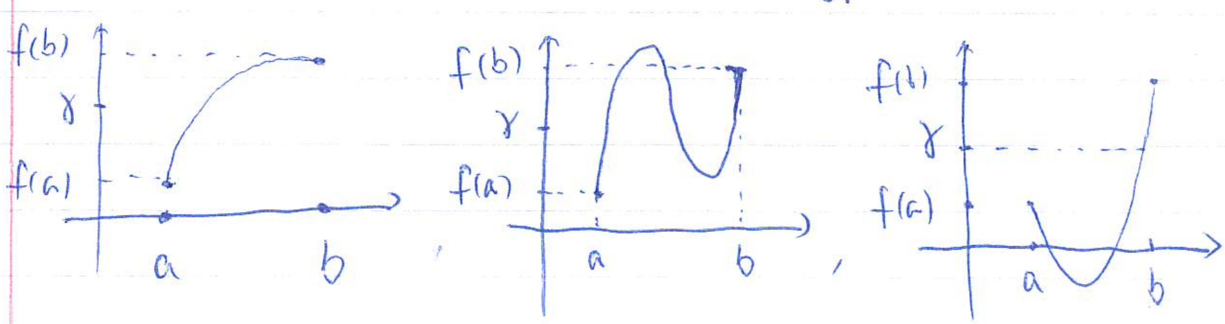
2.4. Intermediate Values

The main goal of this section is to prove the following Intermediate Value Theorem.

Theorem 2.9. Let $f: I \to \mathbb{R}$ be a continuous function defined on a closed, bounded interval $I = [a,b]$. Assume $f(a) \neq f(b)$ and let γ be an any number between $f(a)$ & $f(b)$. Then there exists at least one point $c \in (a,b)$ s.t. $f(c) = \gamma$.

Remark: ① Theorem 2.9 basically says that a continuous function f may not go from $f(a)$ to $f(b)$ without passing through each intermediate values between $f(a)$ & $f(b)$. This certainly matches the geometric vision of continuity

or



② Note that if we take any x_1 & x_2 s.t.

$a < x_1 < x_2 < b$,

then f is continuous function on the smaller interval $[x_1, x_2]$. Thus we can apply Thm 2.9 to $f: [x_1, x_2] \to \mathbb{R}$ and obtain:

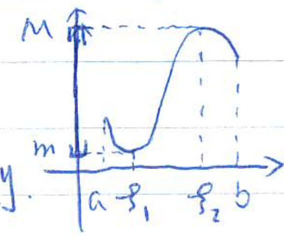
if $f(x_1) \neq f(x_2)$, then $f(x)$ must assume all values between $f(x_1)$ and $f(x_2)$

③ In particular, for f as in Thm 2.9, by Theorem 2.8, f has at least one minimum point ξ_1 where f assumes its minimum m & has at least one maximum point ξ_2 where f assumes its maximum M, i.e.

$f(\xi_1) = m$ is the minimum value of f on I

$f(\xi_2) = M$ is the maximum value of f on I

Then by Thm 2.9, f must assume all values between m & M. Thus $f(I) = [m, M]$, i.e. a closed, bounded interval.



We may formulate it as a corollary.

Corollary: Let $f: I \to \mathbb{R}$ be continuous on $I = [a,b]$. Then $f(I) = [m, M]$ where m is the minimum of f & M is the maximum of f.

Proof of Thm 2.9: Without loss of generality, we assume $f(a) < f(b)$. Set $I_1 = I = [a,b]$. Divide I_1 into two sub-intervals with equal length by introducing its mid-point $c_1 = \frac{a+b}{2}$ Take any $f(a) < \gamma < f(b)$

If $f(c_1) = \gamma$, then we are done; otherwise:

case (I): if $f(c_1) < \gamma$, then we set $I_2 = [c_1, b]$,

Thus $f(c_1) < \gamma < f(b)$

case (II): if $f(c_1) > \gamma$, then we set $I_2 = [a, c_2]$.

Thus $f(a) < \gamma < f(c_1)$

In any case, we have $f($left end-pt of $I_2) < \gamma < f($right end-pt of $I_2)$

Then we repeat the same process for I_2 by introducing its mid-pt. c_2. Then either

$f(c_2) = \gamma$, we are done;

or we obtain I_3 which is half of I_2 s.t.

$f($left end-pt of $I_3) < \gamma < f($right end pt of $I_3)$

By induction, either we find at step n a c_n with $a < c_n < b$ s.t. $f(c_n) = \gamma$; or we construct a nest

$$I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n \supset \cdots$$

s.t.: if we set $I_n = [a_n, b_n]$, $\forall n \geq 1$, then

(i) $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \quad \forall n \geq 1$

(ii) $b_n - a_n = \frac{b-a}{2^{n-1}}$

(iii) $f(a_n) < \gamma < f(b_n)$, $\forall n \geq 1$.

By Theorem 1.15 and its proof regarding nest, we have $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$ and $\lim_{n\to\infty} a_n = \xi = \lim_{n\to\infty} b_n$

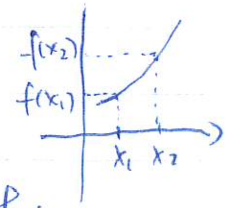
By continuity of f and Thm 2.2, we have

$$\lim_{n\to\infty} f(a_n) = f(\xi) = \lim_{n\to\infty} f(b_n)$$

By squeeze thm & (iii) above, we must have $\gamma = f(\xi)$. Clearly $\xi \in I$ & $\xi \neq a, b$ since $f(a) < \gamma = f(\xi) < f(b)$. Thus $\xi \in (a, b)$ & $f(\xi) = \gamma$ as desired. $\square$

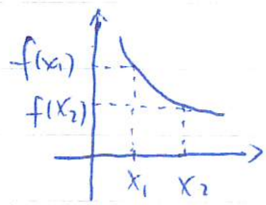
2.5 Monotone Functions and Inverse Functions

Definition 2.6. Let $f: I \to \mathbb{R}$ be a function. We say f is



(I) monotone increasing on I if:
$f(x_1) \le f(x_2) \quad \forall x_1 < x_2 \in I$

(II) strictly monotone increasing on I if:
$f(x_1) < f(x_2) \quad \forall x_1 < x_2 \in I$



(III) monotone decreasing on I if:
$f(x_1) \ge f(x_2) \quad \forall x_1 < x_2 \in I$

(IV) Strictly monotone decreasing on I if:
$f(x_1) > f(x_2) \quad \forall x_1 < x_2 \in I$

A (strictly) monotone function is a function that is either (strictly) monotone increasing or (strictly) monotone decreasing.

Definition 2.7. Consider a function $f: I \to \mathbb{R}$. We say f is 1-to-1 if $f(x_1) \ne f(x_2) \ \forall x_1 \ne x_2 \in I$

Let $J = f(I)$. Then $\forall y \in J$, $\exists$ a unique $x \in I$ s.t.
$$f(x) = y$$
Thus we may define a new function $g: J \to I$

such that $g(y) = x$, where $f(x) = y$. (*)
Such function g is called the inverse of f, denoted $g = f^{-1}$. By (*), we have

$g(f(x)) = g(y) = x \Rightarrow g \circ f : I \to I$ is the identity map

$f(g(y)) = f(x) = y \Rightarrow f \circ g : J \to J$ is the identity map as well.

Lemma: If $f: I \to \mathbb{R}$ is strictly monotone, then f is 1-to-1. Moreover, if f is strictly monotone increasing (decreasing, resp.) then $g = f^{-1}$ is strictly monotone increasing (decreasing, resp.)

Proof: We consider the case where f is strictly monotone increasing. The case where f is decreasing is similar.

By definition, $f(x_1) < f(x_2)$ $\forall x_1 < x_2 \in I$.

Take any $x_1 \neq x_2$, then either $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
or $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$

In both cases, $f(x_1) \neq f(x_2) \Rightarrow f$ is 1-to-1.

$\Rightarrow g = f^{-1}$ exists

Take any $y_1 < y_2 \in J = f(I)$. Then there are x_1 & $x_2 \in I$ s.t. $f(x_1) = y_1$ & $f(x_2) = y_2$

Then either $x_1 > x_2$ or $x_1 < x_2$. But $x_1 > x_2$ implies $y_1 > y_2$. Thus $x_1 < x_2$. By definition

$$x_1 = g(y_1) < g(y_2) = x_2 \quad \forall\ y_1 < y_2 \in J$$

$\Rightarrow$ g is strictly monotone increasing.

Theorem 2.10: Let f be a strictly monotone, continuous function on a closed, bounded interval $I = [a, b]$. Then its inverse $g = f^{-1}$ is strictly monotone and continuous on $J = f(I)$.

Proof: Again we assume that f is strictly montone increasing. The other case can be done similarly.

Then by Corollary of Section 2.4, we know that $J = f(I) = [f(a), f(b)]$. By Lemma, we know that $g = f^{-1}: [f(a), f(b)]$ is strictly monotone increasing. Thus to prove Thm 2.10, we only need to show that g is continuous on $[f(a), f(b)]$.

So we need to show that g is continuous at every $\gamma \in [f(a), f(b)]$. We shall focus on the case $\gamma \in (f(a), f(b))$. The cases with end-points can be done similar and will be left as a homework ~~to~~ problem.

From now on, we fix a $\gamma \in (f(a), f(b))$ & will show g is continuous at γ. First we note that $g(\gamma) = \xi \in (a,b)$. Note $f(\xi) = \gamma$.

By definition of continuity, we need to show: $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $|g(y) - g(\gamma)| < \varepsilon$

$$\forall\ |y-\gamma| < \delta \ \&\ y \in J.$$

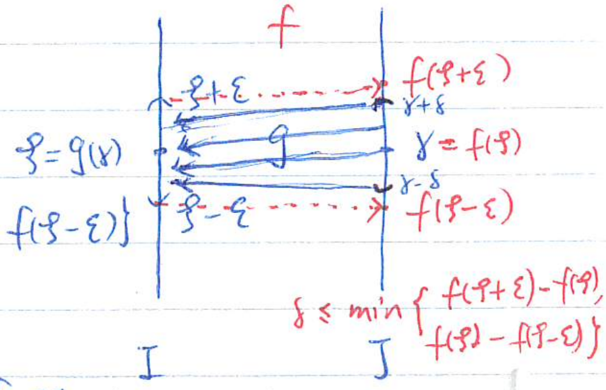
Note we only need to consider very small ε. Indeed in the definition, if δ works for ε, then it works for all $\varepsilon' > \varepsilon$.

Thus we fix any $\varepsilon > 0$ s.t. $[\xi-\varepsilon, \xi+\varepsilon] \subset I$.

Since $\xi - \varepsilon < \xi < \xi + \varepsilon \in I$

$\Rightarrow f(\xi-\varepsilon) < f(\xi) < f(\xi+\varepsilon)$

Set $\delta = \min\{f(\xi+\varepsilon) - f(\xi),\ f(\xi) - f(\xi-\varepsilon)\}$



Then

$$f(\xi-\varepsilon) \le \gamma-\delta < \gamma < \gamma+\delta \le f(\xi+\varepsilon) \qquad (*)$$

In particular $(\gamma-\delta, \gamma+\delta) \subset J$. Moreover,

$y \in (\gamma-\delta, \gamma+\delta) \iff |y-\gamma| < \delta \iff \gamma-\delta < y < \gamma+\delta$

By monotonicity of g, we have

$$g(f(\xi-\varepsilon)) \le g(\gamma-\delta) < g(y) < g(\gamma+\delta) \le g(f(\xi+\varepsilon))$$

$$\Rightarrow \xi-\varepsilon < g(y) < \xi+\varepsilon$$

$$\Leftrightarrow |g(y)-\xi| < \varepsilon \Leftrightarrow |g(y)-g(\gamma)| < \varepsilon.$$

To sum up, $\forall \varepsilon > 0$ small, we've found a $\delta > 0$ s.t. $|g(y)-g(\gamma)| < \varepsilon \quad \forall\ |y-\gamma| < \delta$

$\Rightarrow g$ is continuous at γ. $\square$

Question: Where did we use the continuity of f?

Examples

① Consider $f(x)=x^2 : [0,+\infty) \to [0,+\infty)$. It's inverse $g : [0,+\infty) \to [0,+\infty)$ is continuous & strictly monotone increasing.

Proof: Clearly, $f(x)=x^2$ is continuous and strictly monotone increasing on $[0,+\infty)$. Moreover $\forall y \in [0,+\infty)$, there is a ~~unique~~ $n \in \mathbb{Z}^+$ s.t. $n = m^2 > y$

Thus $y \in [0,\cancel{m}^{\,n}] = f([0,m])$ by corollary of Thm 2.9.

$\Rightarrow \exists x \in [0,m]$ s.t. $f(x)=y$. Since y is arbitrarily chosen, we have $f([0,+\infty)) = [0,+\infty)$. Thus the inverse $g = f^{-1}$ is defined on $[0,+\infty)$.

To show g is strictly monotone increasing, take any $y_1 < y_2 \in [0,+\infty)$. Then we may consider $g: [0, M] \to [0, g(M)]$ for some $M > y_2$. Then $g|_{[0,M]}$ is the inverse of $f|_{[0,g(M)]}$. Thus Thm 2.10 implies that g is strictly monotone increasing on $[0,M] \Rightarrow g(y_1) < g(y_2)$.

Since $y_1 < y_2$ are abitrarily chosen $\Rightarrow g$ is strictly monotone increasing on $[0,+\infty)$.

To show g is continuous on $[0,+\infty)$. Take any $\gamma \in [0,+\infty)$. We may again restrict g to $[0, M]$ for some $M > \gamma$. Then $g|_{[0,M]}$ is the inverse of the continuous function $f|_{[0,g(M)]} \Rightarrow g$ is continuous on $[0, M]$ $\Rightarrow g$ is continuous at $\forall \gamma \Rightarrow g$ is continuous on $[0,+\infty)$ since γ is abitrarily chosen.

We denote $g(x) = \sqrt{x}$ $\forall x \in [0,+\infty)$.

② Through the same process, we can show for any $n \in \mathbb{Z}_+$, there is a continuous & strictly monotone increasing function

$$g : [0,+\infty) \to [0,+\infty)$$ which inverts

$f : [0,+\infty) \to [0,+\infty)$ where $f(x) = x^n$.

We denote $g(x) = x^{\frac{1}{n}}$.

③. By ② we may define $\forall\, n, m \in \mathbb{Q}_+$, $\forall x \in \mathbb{R}_+ \cup \{0\}$

$$x^{\frac{m}{n}} = \underbrace{x^{\frac{1}{n}} \cdot x^{\frac{1}{n}} \cdots x^{\frac{1}{n}}}_{m \text{ times}}.$$ Thus

$g(x) = x^{\frac{m}{n}} : [0,+\infty) \to [0,+\infty)$ is continuous & strictly monotone increasing, i.e. $g(x) = x^r$ is a well-defined cont. & strictly monotone increasing function from $[0,+\infty)$ to $[0,+\infty)$.

④ If $r \in \mathbb{Q}_-$, i.e. a negative rational number. we define $g(x) = x^r := \frac{1}{x^{-r}}$ which is well-define $\forall\, x > 0$ since $-r \in \mathbb{Q}_+$. Since x^{-r} is continuous & increasing on $(0,+\infty)$

$\Rightarrow g(x) = \frac{1}{x^{-r}} : (0,+\infty) \to (0,+\infty)$ is cont. & decreasing.

Now we've defined $g(x) = x^r$ for all $r \in \mathbb{Q}$. and have shown their continuity on their domain.

⑤ We know that $\overset{f(x)}{\sin(x)} : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is strictly monotone & increasing & continuous.

Thus it has an inverse

$$\sin^{-1}(x) : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$$

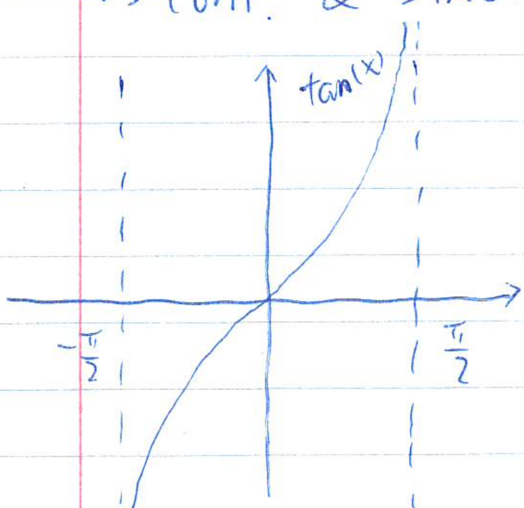
sometimes $\sin^{-1}(x) = \arcsin(x)$

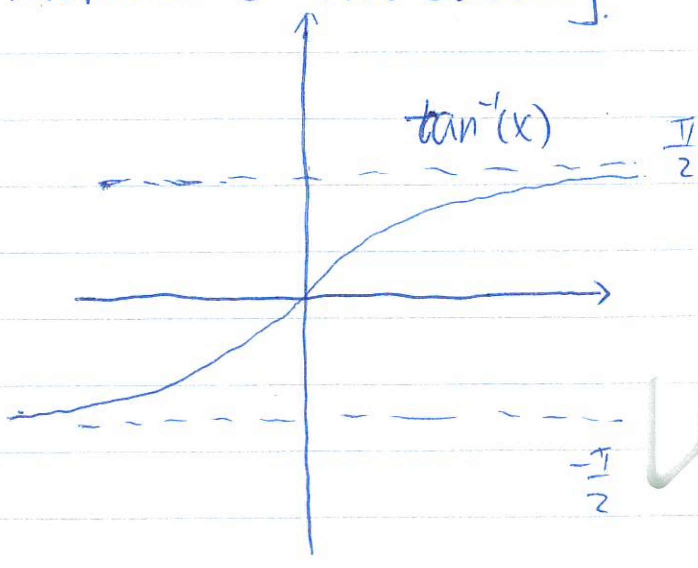
which is again cont. & strictly monotone increasing.

⑥ We know that $f(x) = \tan(x) : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is cont. & strictly monotone increasing.

Thus $\tan^{-1}(x) : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is cont. & strictly monotone increasing.

sometimes $\tan^{-1}(x) = \arctan(x)$





2.6 Exponentials & Logarithms

The goal of this section is to define and to show continuity of

- $f(x) = a^x$, $a>0$, $a \neq 1$ &
- $f(x) = \log_a(x)$, $a>0$, $a\neq 1$.

$a=1$ is trivial since $1^x = 1$ $\forall x$

Since $\log_a(x)$ will be defined as inverse function of a^x, we shall mainly focus on

$$f(x) = a^x, \quad a>0, \quad a \neq 1.$$

Note by example ④ of section 2.5, we've defined a^r for any $r \in \mathbb{Q}$. Thus we only need to extend the domain from $\mathbb{Q}$ to $\mathbb{R}$. First, we note that $\forall n, m \in \mathbb{Z}_+$, it holds that

$$\left.\begin{aligned} a^m \cdot a^n &= a^{m+n} \\ (a^m)^n &= (a^m)^n \end{aligned}\right\} \qquad (1)$$

One can easily check by definition of a^n, $n \in \mathbb{Z}$.

First we want to extend (1) to $\mathbb{Q}$, i.e.

<u>Lemma 1</u>: $\forall \alpha, \beta \in \mathbb{Q}$, & $\forall a>0$, we have

$$a^\alpha \cdot a^\beta = a^{\alpha+\beta}, \quad (a^\alpha)^\beta = (a^\alpha)^\beta \qquad (2).$$

Proof: We may write $\alpha = \frac{p}{q}$ & $\beta = \frac{m}{n}$, where $p, q, m, \& n \in \mathbb{Z}_+$. Then

$$a^{\alpha} \cdot a^{\beta} = a^{\frac{p}{q}} \cdot a^{\frac{m}{n}} = a^{\frac{pn}{qn}} \cdot a^{\frac{mq}{nq}}$$

$$= (a^{\frac{1}{qn}})^{pn} \cdot (a^{\frac{1}{qn}})^{mq}$$

by (1)

$$= (a^{\frac{1}{qn}})^{pn+mq}$$

by def from ~~exp~~ example (3) (4) of sec. 2.5.

$$= a^{\frac{pn+mq}{qn}}$$

$$= a^{\frac{p}{q}+\frac{m}{n}} = a^{\alpha+\beta}$$ done with the first one.

For $(a^{\alpha})^{\beta} = a^{\alpha\cdot\beta}$, we first consider the case where $\alpha = \frac{1}{q}$ & $\beta = \frac{1}{n}$. Then

$(a^{\alpha})^{\beta} = (a^{\frac{1}{q}})^{\frac{1}{n}}$. By definition of $a^{\frac{1}{m}}$ & (1), we have $[(a^{\alpha})^{\beta}]^{n\cdot q} = [(a^{\frac{1}{q}})^{\frac{1}{n}}]^{nq} = ([(a^{\frac{1}{q}})^{\frac{1}{n}}]^{n})^{q}$

$= (a^{\frac{1}{q}})^{q} = a$.

On the other hand, $(a^{\alpha\beta})^{nq} = (a^{\frac{1}{qn}})^{qn} = a$

$\Rightarrow (a^{\alpha})^{\beta} = a^{\alpha\beta}$ (*) since both are the positive solution of $x^{nq} = a$.

Now for general $\alpha = \frac{p}{q}$ & $\beta = \frac{m}{n}$, we have

$$(a^{\alpha})^{\beta} = (a^{\frac{p}{q}})^{\frac{m}{n}} \overset{def}{=} [((a^{p})^{\frac{1}{q}})^{\frac{1}{n}}]^{m} \overset{(*)}{=} [(a^{p})^{\frac{1}{qn}}]^{m} \underset{def}{=} (a^{\frac{p}{qn}})^{m}$$

$$= [(a^{\frac{1}{qn}})^{p}]^{m} \underset{(1)}{=} (a^{\frac{1}{qn}})^{pm} \overset{def}{=} a^{\frac{pm}{qn}} = a^{\alpha\cdot\beta}$$ □

We are almost ready to define a^x for $x \notin \mathbb{Q}$. We still need two more lemmas.

Lemma 2: Let $a>1$ and consider $f(r)=a^r$ as a function on $\mathbb{Q}$. Then f is strictly monotone increasing.

Proof: First we show $a^{\frac{1}{n}}>1$ $\forall n \in \mathbb{Z}_+$. Indeed, if $a^{\frac{1}{n}} \le 1$, then $(a^{\frac{1}{n}})^n \le 1^n \Rightarrow a \le 1$, contradicts with $a>1$.

As a consequence, we have $a^r>1$ $\forall r \in \mathbb{Q}_+$. Indeed, we may write $r=\frac{m}{n}$ where $m, n \in \mathbb{Z}_+$. Then $a^r = a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} > 1$ since $a^m > 1$.

Now take any $r>s \in \mathbb{Q}$. Then $r-s \in \mathbb{Q}_+$. Thus $a^{r-s}>1 \Rightarrow (a^{r-s}) \cdot a^s > a^s \overset{\text{Lemma 1}}{\Rightarrow} a^{r-s+s} > a^s$

$$\Rightarrow a^r > a^s$$

$\Rightarrow f$ is ~~m~~ strictly monotone increasing on $\mathbb{Q}$. □

Lemma 3: Between any two real numbers, there is a rational number.

Proof: Let $\alpha < \beta$ be any two real numbers.

If there is an ~~integers~~ integer m s.t.

$\alpha < m < \beta$.

then we are done as $m \in \mathbb{Q}$.

Otherwise, there is no integer bewteen α & β.

Then we must have that

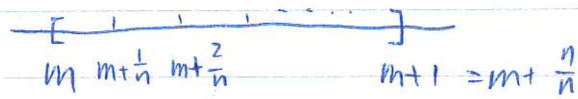
$$m \leq \alpha < \beta \leq m+1 \quad \text{for some } m \in \mathbb{Z}.$$

Let $\varepsilon = \beta - \alpha > 0$. Then we can find a $n \in \mathbb{Z}_+$ s.t.

$\frac{1}{n} < \varepsilon$.

We divide the interval $[m, m+1]$ into n intervals with length $\frac{1}{n}$ as $[m+\frac{j}{n}, m+\frac{j+1}{n}]$, where

$j = 0, 1, \cdots, n-1$



Since $\frac{1}{n} < \beta - \alpha$, we must have $j_0 \in \{1, \cdots, n-1\}$ s.t.

$$\alpha < m + \frac{j_0}{n} < \beta.$$

Indeed, if there is no such j_0, then there is another $j' \in \{0, \cdots, n-1\}$ s.t. $\quad m + \frac{j'}{n} \leq \alpha < \beta \leq m + \frac{j'+1}{n}$

$$\Rightarrow \beta - \alpha \leq m + \frac{j'+1}{n} - \left(m + \frac{j'}{n}\right) = \frac{1}{n} \quad \text{contraditison}.$$

Since $m + \frac{j_0}{n} \in \mathbb{Q}$, we are done. $\square$

Corollary: $\mathbb{Q}' = \mathbb{R}$. (Homework problem, try to show this by definition of limit pt)

In fact, we have something stronger.

Lemma 4: For any $x \in \mathbb{R}$, we can find a strictly monotone decreasing sequence of rational numbers $\{r_n\}_{n\geq 1}$ s.t.

$$\lim_{n\to\infty} r_n = x$$

Proof: By Lemma 3, there is a $r_1 \in \mathbb{Q}$ s.t.

$$x < r_1 < x+1.$$

By Lemma 3 again, there is a $r_2 \in \mathbb{Q}$ s.t.

$$x < r_2 < x + \min\{r_1 - x, \tfrac{1}{2}\}$$

$\Rightarrow$ $x < r_2 < x + r_1 - x = r_1$ & $x < r_2 < x + \frac{1}{2}$

By induction, we can find a sequence $\{r_n\}_{n\geq 1}$ of rational numbers r_n s.t.

$$x < r_{n+1} < r_n \quad \& \quad x < r_{n+1} < x + \frac{1}{n+1}$$

$\Rightarrow$ $\lim_{n\to\infty} r_n = x$ & r_n is strictly monotone decreasing. For simplicity, we denote it as

$$r_n \downarrow x \quad \text{as } n\to\infty.$$

Lemma 5: Fix $a>1$ & $x \notin \mathbb{Q}$ & choose $\{r_n\}_{n\geq 1}$ in $\mathbb{Q}$ s.t. $r_n \downarrow x$ as $n \to \infty$. Then the sequence $\{a^{r_n}\}_{n\geq 1}$ is monotone decreasing and bounded below.

Proof: By Lemma 2 & the fact $\{r_n\}$ is strictly monotone decreasing, we get $\{a^{r_n}\}$ is strictly monotone decreasing. Take any $s<x$ & $s \in \mathbb{Q}$. Then $\forall n \in \mathbb{Z}_+$, we have

$s<x<r_n \Rightarrow a^s < a^{r_n} \Rightarrow \{a^{r_n}\}$ is bounded below by a^s. □

By Lemma 5, $\lim\limits_{n\to\infty} a^{r_n}$ exists.

Definition 2.8: ① Fix $a>1$ & $x \notin \mathbb{Q}$, we define

(*) $\underline{a^x = \lim\limits_{n\to\infty} a^{r_n}}$ where $\{r_n\}_{n\geq 1}$ in $\mathbb{Q}$ s.t. $r_n \downarrow x$.

② Now we can define $f(x) = a^x : \mathbb{R} \to \mathbb{R}$.

Lemma 6: The definition (*) is independent of the choice of such $\{r_n\}_{n\geq 1}$. In other words if $\{s_n\}_{n\geq 1}$ in $\mathbb{Q}$ s.t. $s_n \downarrow x$, then $\lim\limits_{n\to\infty} a^{s_n} = \lim\limits_{n\to\infty} a^{r_n}$.

Proof: Fix $\forall n \in \mathbb{Z}_+$. Since $s_n > x$ & $\lim r_m = x$, we have $r_m < s_n$ $\forall m$ large. Indeed, by definition of convergent sequence, for $\varepsilon = s_n - x > 0$ $\exists m_0$ s.t. $|r_m - x| < \varepsilon$ $\forall m \geq m_0$

$$\Rightarrow r_m - x < \varepsilon \Rightarrow r_m < x + \varepsilon = x + s_n - x \quad \forall m \geq m_0$$

$$\Rightarrow r_m < s_n \quad \forall m \geq m_0 \xRightarrow{\text{Lemma 2}} a^{r_m} < a^{s_n} \quad \forall m \geq m_0$$

By Thm 1.10, we have $\lim\limits_{m\to\infty} a^{r_m} < a^{s_n}$. $\forall n \in \mathbb{Z}_+$

Again by Thm 1.10, $\lim\limits_{m\to\infty} a^{r_m} \leq \lim\limits_{n\to\infty} a^{s_n}$.

Switching the role of $\{r_m\}$ & $\{s_n\}$, we obtain

$$\lim_{n\to\infty} a^{s_n} \leq \lim_{m\to\infty} a^{r_m}$$

$\Rightarrow \lim\limits_{n\to\infty} a^{s_n} = \lim\limits_{n\to\infty} a^{r_m}$ for any sequences of rational numbers $\{s_n\}$ & $\{r_m\}$ s.t. $s_n \downarrow x$, $r_n \downarrow x$.

$\square$

Now we want explore properties of $f(x) = a^x$.

Theorem 2.11: $\forall a > 1$, $f(x) = a^x : \mathbb{R} \to \mathbb{R}$ is strictly monotone increasing.

Proof: Fix any $x_1 < x_2$ in $\mathbb{R}$. First we assume $x_1, x_2 \in \mathbb{Q}$.

Then by Lemma 3, we can find two rational numbers r & s s.t.

$$x_1 < r < s < x_2$$

Taking $r_n \downarrow x_1$ & $s_n \downarrow x_2$. Then for all large n, we have $x_1 < r_n < r < s < x_2 < s_n$

$\Rightarrow \quad a^{r_n} < a^r < a^s < a^{s_n} \quad \forall n$ large

$\Rightarrow \quad \lim_{n\to\infty} a^{r_n} \le a^r < a^s \le \lim_{n\to\infty} a^{s_n}$

$\Rightarrow \quad a^{x_1} < a^{x_2}$

If only one of x_1 & x_2 is rational, say $x_1 \in \mathbb{Q}$ & $x_2 \notin \mathbb{Q}$.

The argument above can be modified to be

$$a^{x_1} < a^r < a^{s_n} \quad \forall n \ge 1$$

$\Rightarrow a^{x_1} < a^r \le \lim_{n\to\infty} a^{s_n} \Rightarrow a^{x_1} < a^{x_2}$.

The case of $x_1 \notin \mathbb{Q}$ & $x_2 \in \mathbb{Q}$ can be done similarly. The case of x_1 & $x_2 \in \mathbb{Q}$ has been taken care of in Lemma 2. Thus we are done. $\square$

Clearly $f(x) = a^x > 0$ $\forall x \in \mathbb{R}$. Indeed, $a^r > 0$ $\forall r \in \mathbb{Q}$ is clearly. If $x \notin \mathbb{Q}$, then $a^x > a^r > 0$ for any $r \in \mathbb{Q}$ s.t. $r < x$. Thus $f(\mathbb{R}) \subset (0, +\infty)$.

<u>Lemma 7.</u>: $a^{\alpha+\beta} = a^\alpha \cdot a^\beta$
$(a^\alpha)^\beta = a^{\alpha \cdot \beta}$ $\forall \alpha, \beta \in \mathbb{R}$.

Proof: We've done the case for $\alpha, \beta \in \mathbb{Q}$ in Lemma 1. Now if only one of them is irrational, say $\alpha \notin \mathbb{Q}$, $\beta \in \mathbb{Q}$, then we can find $r_n \downarrow \alpha$.

Then $r_n + \beta \downarrow \alpha + \beta$. Thus we have

$$a^{\alpha+\beta} = \lim_{n\to\infty} a^{r_n+\beta} \overset{\text{Lemma 1}}{=\!=} \lim_{n\to\infty} (a^{r_n} \cdot a^\beta)$$

$$\underset{\text{L.L.}}{\overset{\text{Product}}{=\!=}} a^\beta \cdot \lim_{n\to\infty} a^{r_n} = a^\beta \cdot a^\alpha$$

If both α & β are irrational, we take $r_n \downarrow \alpha$ & $s_n \downarrow \beta$. Then $r_n + s_n \downarrow \alpha + \beta$

Thus $$a^{\alpha+\beta} = \lim_{n\to\infty} a^{r_n+s_n} \overset{\text{Lemm 1}}{=\!=} \lim_{n\to\infty} (a^{r_n} \cdot a^{s_n})$$

$$\underset{\text{L.L.}}{\overset{\text{Product}}{=\!=}} \lim_{n\to\infty} a^{r_n} \cdot \lim_{n\to\infty} a^{s_n} = a^\alpha \cdot a^\beta$$

The second formula can be done similarly □

Theorem 2.12. $\forall a>1$, $f(x)=a^x : \mathbb{R} \to \mathbb{R}_+$ is continuous.

Proof: First, we claim that we only need to show continuity of f at $x=0$. Indeed, if $\lim_{x\to 0} f(x) = f(0) = a^0 = 1$. (i.e. continuity at 0), then $\forall x_0 \in \mathbb{R}$, we have

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} a^x \overset{\text{Lemma 7}}{=} \lim_{x-x_0\to 0} a^{x-x_0}\cdot a^{x_0}$$

$$\overset{\text{Thm 2.4}}{=} a^{x_0}\cdot \lim_{x-x_0\to 0} a^{x-x_0} \underset{y=x-x_0}{=} a^{x_0} \lim_{y\to 0} a^y = a^{x_0} = f(x_0).$$

Thus we only need to show continuity at 0.

To show $\lim_{x\to 0} a^x = 1$, we fix any $\varepsilon > 0$.

Then we may pick a $\delta = \frac{1}{n_0} > 0$ where n_0 is so large s.t. $1-\varepsilon < a^{-\frac{1}{n_0}} < a^{\frac{1}{n_0}} < 1+\varepsilon$

Here we use $\cancel{\lim_{n\to\infty} a^{\frac{1}{n}} =}$ $(1+\varepsilon)^n > a$ $\forall n$ large
& $(1-\varepsilon)^n < a^{-1}$ $\forall n$ large

Now $\forall x$ s.t. $|x|<\delta \iff -\delta < x < \delta \iff -\frac{1}{n_0} < x < \frac{1}{n_0}$

$\overset{\text{Thm 2.11}}{\Longrightarrow} 1-\varepsilon < a^{-\frac{1}{n_0}} < a^x < a^{\frac{1}{n_0}} < 1+\varepsilon$

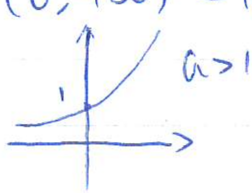
$\Rightarrow |a^x - 1| < \varepsilon$ $\forall |x|<\delta \Rightarrow \lim_{x\to 0} a^x = 1$. $\square$

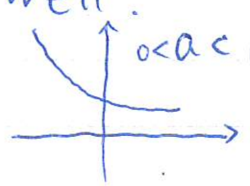
Lemma 8. Consider $f(x) = a^x : \mathbb{R} \to \mathbb{R}$, $a > 1$. Then
$$f(\mathbb{R}) = \mathbb{R}_+ = (0, +\infty)$$

Proof: Note $f(n) = a^n > 1 + n(a-1)$ (binomial expansion) which can be arbitrarily large. On the other hand, $f(-n) = a^{-n} = \left(\frac{1}{a}\right)^n \to 0$ as $n \to \infty$. Thus,
$$\forall y \in (0, +\infty),\ \exists n \in \mathbb{Z}_+ \text{ s.t. } y \in [f(-n), f(n)]$$
By Thm 2.11, 2.12 & Corollary of Sec 2.5, we have
$$y \in [f(-n), f(n)] = f([-n, n]) \Rightarrow y = f(x) \text{ for some } x \in [-n, n]$$
$$\Rightarrow f(\mathbb{R}) = (0, +\infty) \text{ since } \overset{>0}{y} \text{ is arbitrary.} \qquad \square$$

- Now we've got almost everything for $f(x) = a^x$ when $a > 1$: strictly monotone increasing, continuity, and $f(\mathbb{R}) = (0, +\infty) = \mathbb{R}_+$.

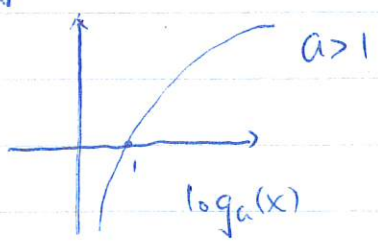
- For $0 < a < 1$, we define $f(x) = a^x := \left(\frac{1}{a}\right)^{-x}$ which is well-defined since $\frac{1}{a} > 0$. Clearly, f is continuous since it's a composition of $g(x) = \left(\frac{1}{a}\right)^x$ & $h(x) = -x$ which are continuous. Moreover, f is strictly monotone decreasing (homework problem). Finally, $f(\mathbb{R}) = g(\mathbb{R}) = (0, +\infty) = \mathbb{R}_+$ as well.

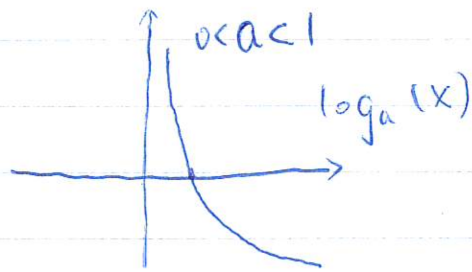




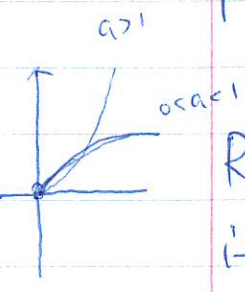
Definition 2.9: Fix $a>0$, $a\neq 1$, $f_a(x)=a^x$ is strictly monotone & continuous with $f(\mathbb{R})=\mathbb{R}_+$. Thus it has an inverse $f_a^{-1}:\mathbb{R}^+\to\mathbb{R}$ which is strictly monotone (increasing when $a>1$ & decreasing when $0<a<1$) and continuous. We denote it by $f_a^{-1}(x)=\log_a(x)$. Thus $\log_a a^x = x$, $a^{\log_a x}=x$.

In particular, if $a=e=\lim\limits_{n\to\infty}(1+\frac{1}{n})^n$, we denote it by $\log_e(x)=\ln(x)$

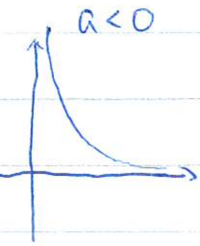




Definition 2.10: Fix $a\notin\mathbb{Q}$, we define the power function $P_a(x)=x^a := e^{a\ln(x)}:\mathbb{R}^+\to\mathbb{R}$.



Remark: ① Note if $a\in\mathbb{Q}$, then $x^a=e^{a\ln(x)}$ since it's not difficult to see $\ln(x^a)=a\ln(x)$ $\forall a\in\mathbb{Q}$. Thus the definition matches with the previous definition when $a\in\mathbb{Q}$.



② Clearly, $P_a(x)=x^a$ is continuous since it's a composition of continuous functions

③ $P_a(x)$ is strictly monotone (increasing if $a>0$; decreasing if $a<0$). HW problem.