1.2 Limits of Sequences

1.2.1 Convergent Sequences

Definition 1.2. A sequence is a map from $\mathbb{Z}_+$ to $\mathbb{R}$, denoted

$$a : \mathbb{Z}_+ \longrightarrow \mathbb{R}$$
$$n \longmapsto a(n).$$

Here $\mathbb{Z}_+$ is the domain of a. The range of the sequence is the set of all distinct values $a(n)$.

The terms of a sequence are the real numbers

$$a(1), a(2), \cdots, a(n), \cdots$$

which are usually denoted ~~as~~ with sub-scripts

$$a_1, a_2, \cdots, a_n, \cdots.$$

In particular, a_n is the nth term of the sequence.

A sequence is usually denoted by $\{a_n\}_{n\geq 1}$, or simply $\{a_n\}$.

Example 1.

① $a : \mathbb{Z}_+ \to \mathbb{R}$, $a(n) \equiv c$ (meaning $a(n) = c$ for all n) is called a constant sequence. Although the sequence must be listed as $c, c, \cdots, c, \cdots$, the range of this sequence is a single point set $\{c\}$.

② $\{a_n = \frac{1}{n}\}_{n\geq 1}$ is a sequence with range $\{\frac{1}{n} : n \geq 1\}$.

③ $\{a_n = n\}_{n\geq 1}$ is a sequence with range $\{n : n\geq 1\} = \mathbb{Z}_+$.

In this case, we say $\{a_n\}$ is convergent.

Definition 1.3 We say a sequence $\{a_n\}$ converges to L as n goes to infinity if the following holds true:

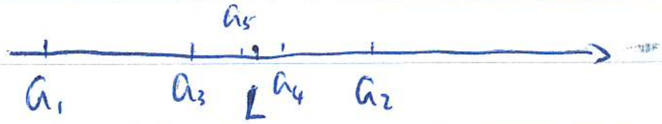
for any $\varepsilon>0$, there exists an positive integer n_0 (i.e. $n_0 \in \mathbb{Z}_+$), s.t.

$$|a_n - L| < \varepsilon \quad \text{for all} \quad n \geq n_0$$

L is called the limit of $\{a_n\}$ & denoted $\lim_{n\to\infty} a_n = L$

Again, this is the most difficult notion of college math & we shall spend a lot of time understanding it. But first let's try to describe it geometrically. Let's put everything on the real line.

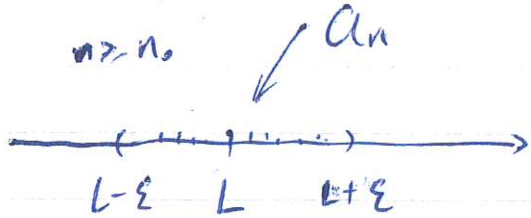
Roughly speaking, $\lim_{n\to\infty} a_n = L$ means that as $n\to\infty$, a_n tends to L



It describes the asymptotic behavior of a_n as $n\to\infty$.

But what do we really mean by saying a_n tends to L as $n\to\infty$? Precisely, it means that no matter how small $\varepsilon>0$ is, eventually, all a_n are within ε-distance to L, i.e. $|a_n-L|<\varepsilon$. What the rigorous meaning of "eventually all"? It means there is a $n_0\geq 1$, that $|a_n-L|<\varepsilon$ for all

$n \geq n_0$.



Remark: ① From the definition, we may see that $\lim a_n = L$ concerns only the asympototic behavior of $\{a_n\}$. It doesn't concern the first finite number of terms. In other words, if

$$\lim_{n\to\infty} a_n = L$$

then the change of first finite number of terms won't affect this fact.

② The key thing in the definition 1.3 is that the choice of $\varepsilon > 0$ is arbitrary. In particular $\varepsilon > 0$ can be arbitrarily small. In fact, we ~~need~~ only need to care about small ε. Because if

$$|a_n - L| < \varepsilon_1 \quad \text{for } \varepsilon_1 < \varepsilon_2$$

then of course $|a_n - L| < \varepsilon_2$. Or if eventually the distance of a_n & L is less than a small ε_1, then of course they are less than any other ε_2 that is bigger than ε_1.

③ If $\{a_n\}$ does not converge anywhere as $n \to \infty$, then we say that $\{a_n\}$ is divergent.

④ Convergence or divergence of a sequence concerns the asymptotic behavior of the sequence $\{a_n\}_{n\geq 1}$. It's impossible for us to go over every term to determine such behivor as the number of terms are infinitely many. Instead we use a logic lanague to describe it. In fact, there are symbols for logic terms:

for any : $\forall$

there exists : $\exists$

The we may refrase definition 1.3 as

"$\forall \varepsilon > 0, \exists n_0 \in \mathbb{Z}_+$ s.t. $|a_n - L| < \varepsilon \quad \forall n \geq n_0$."

Although, calculus were discovered by Newton in 17th century. The rigorous definition of convergence sequence were given by Bolzano & Weierstrauss in 19th century. Note all rigorous definitions of continuity, differentiability & integrability are based on the definition of convergent sequences.

In other words, people were using calculus without really understand what are calculus for about 200 years. ~~These~~ It partially explain why definition 1.3 is a difficult one.

To better understand definition, we for sure need to have some concrete examples. Here is the second principle of learning analysis or even mathematics:

(II). One should always use concrete examples to help them understand abstract notions or theorems.

But before we consider examples, we first note the following important facts concerning convergent sequence.

Definition 1.4. Let $\{a_n\}$ be a sequence. A sequence $\{b_k\}_{k\geq 1}$ is called a sub-sequence of $\{a_n\}$ if $b_1 = a_{n_1}, \; b_2 = a_{n_2}, \cdots, b_k = a_{n_k}, \cdots$ for a sequence of increasing positive integers

$$n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots$$

We write such a sequence $\{b_k\}_{k\geq 1}$ as $\{a_{n_k}\}_{k\geq 1}$.

Geometrically, it's not so hard to see that if $\lim_{n\to\infty} a_n = L$ then $\lim_{k\to\infty} b_k = L$ for all such sub-sequence. Because $\{b_k\} = \{a_{n_k}\}$ is part of $\{a_n\}$. So if a_n tends L as $n\to\infty$, then of course $b_k = a_{n_k}$ tends L as $k\to\infty$ (or $n_k\to\infty$). This ~~+~~ is the following theorem.

Theorem 1.2. If $\lim_{n\to\infty} a_n = L$, then $\lim_{k\to\infty} a_{n_k} = L$ for all subsequence $\{a_{n_k}\}$ of $\{a_n\}$.

Proof: How to put the geometric picture into rigorous math proof? Recall $\lim_{n\to\infty} a_n = L$ means $\forall \varepsilon > 0$, $\exists n_0$ s.t. $|a_n - L| < \varepsilon$, $\forall n \geq n_0$.

Now let's fix any subsequence $\{a_{n_k}\}$ of $\{a_n\}$. Then ~~+~~ there is a k_0 s.t. $n_{k_0} \geq n_0$. Since the sequence $\{n_k\}_{k\geq 1}$ increases in k, we have

$$n_k \geq n_{k_0} \geq n_0 \quad \text{for all } k \geq k_0$$

which implies $|a_{n_k} - L| < \varepsilon$. for $\forall k \geq k_0$

Thus we've just showed that:
$\forall \varepsilon > 0, \exists k_0$, s.t. $|a_{n_k} - L| < \varepsilon \quad \forall k \geq k_0$

By definition $\lim\limits_{k \to \infty} a_{n_k} = L$. □

<u>Theorem 1.3</u> (uniqueness of limit):
If $\lim\limits_{n\to\infty} a_n = L$ & $\lim\limits_{n\to\infty} a_n = M$, then $L = M$.

In other words, if $\{a_n\}$ is convergent, then they may only converges to an unique limit.

Idea of Proof: Geometrically, if a_n tends to L as $n \to \infty$, it's certainly not possible for a_n to tends to a different M as $n \to \infty$. How to put it into rigorous math proof? We argue by contradiction.

Proof: Assume $L \neq M$.
We may assume that $L < M$. The we can find a small ε s.t. $L + \varepsilon \leq M - \varepsilon$, e.g. any $\varepsilon \leq \frac{M-L}{2}$ will do.

Indeed, $L + \varepsilon \leq L + \frac{M-L}{2} = \frac{M+L}{2} = M - \frac{M-L}{2} \leq M - \varepsilon$
if $\varepsilon \leq \frac{M-L}{2}$.

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By definition $\lim_{n\to\infty} a_n = L$ means

$\forall \varepsilon > 0, \exists n_0$ s.t. $|a_n - L| < \varepsilon, \ \forall n \geq n_0$

In particular for $\varepsilon = \frac{M-L}{2}$, we can find such a n_0. Similarly for the same $\varepsilon = \frac{M-L}{2}$, we can find another $n_1 \in \mathbb{Z}_+$ s.t.

$$|a_n - M| < \varepsilon, \quad \forall\ n \geq n_1$$

since $\lim_{n\to\infty} a_n = M$ as well.

Thus for any $n \geq \max\{n_0, n_1\}$, it holds that

$$\begin{cases} |a_n - L| < \varepsilon \\ |a_n - M| < \varepsilon \end{cases}$$

$$\Rightarrow |L-M| \leq |L - a_n + a_n - M|$$
$$\leq |L - a_n| + |a_n - M|$$
$$< \varepsilon + \varepsilon$$
$$= \frac{M-L}{2} + \frac{M-L}{2}$$
$$= M-L$$
$$= |M-L|$$

i.e. $|M-L| > |M-L|$, contradiction

Thus the assumption $M \neq L$ is false

$\Rightarrow M = L$ $\square$

$A \Longrightarrow B$: "A implies B"
$A \Longleftarrow B$: "B implies B"
$A \Longleftrightarrow B$: "A is equivalent to B"

1.2.2 Examples of convergent/divergent sequences.

Example 2.

① Fix $a>0$. Show that $\lim_{n\to\infty} \frac{1}{n^a} = 0$

Proof: By definition, we need to show that:

$\forall \varepsilon>0$, there is a n_0 s.t.

$$\left|\frac{1}{n^a} - 0\right| < \varepsilon, \quad \forall\, n \geq n_0.$$

The key thing we need to do here is to find the n_0.

How to do this? We work backwards:

$$\left|\frac{1}{n^a} - 0\right| < \varepsilon \iff \frac{1}{n^a} < \varepsilon \quad \left(\text{since } \frac{1}{n^a} > 0\right)$$

$$\iff \frac{1}{\varepsilon} < n^a$$

$$\iff \left(\frac{1}{\varepsilon}\right)^{\frac{1}{a}} < n.$$

Thus if we pick a n_0 s.t. $n_0 > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{a}}$, then by the argument above, we have for all $n \geq n_0 \left(> \left(\frac{1}{\varepsilon}\right)^{\frac{1}{a}}\right)$

$$\left|\frac{1}{n^a} - 0\right| < \varepsilon$$

To sum up, $\forall \varepsilon>0$, $\exists\, n_0 = \left\lceil \left(\frac{1}{\varepsilon}\right)^{\frac{1}{a}} \right\rceil$ s.t.

$$\left|\frac{1}{n^a} - 0\right| < \varepsilon, \quad \forall\, n \geq n_0$$

$\Rightarrow \lim_{n\to\infty} \frac{1}{n^a} = 0$ $\square$

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Corollary: $\lim_{n\to\infty}\frac{1}{n}=0$, $\lim_{n\to\infty}\frac{1}{n^2}=0$

$\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$, $\lim_{n\to\infty}\frac{1}{n^k}$ (for any $k\in\mathbb{Z}_+$).

② Fix $0<a<1$. Show that $\lim_{n\to\infty} a^n = 0$

Proof: Here we need the formula of binomial expansion: $(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$

$\binom{n}{k}=\frac{n\cdot(n-1)\cdots(n-k+1)}{k\cdot(k-1)\cdots 1}$

$\binom{n}{0}=\binom{n}{n}=1$

$\binom{n}{1}=\frac{n}{1}=n$

$\binom{n}{2}=\frac{n(n-1)}{2}$

$$= x^n+nx^{n-1}\cdot y+\frac{n(n-1)}{2}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3\cdot2\cdot1}x^{n-3}\cdot y^3+\cdots+ nxy^{n-1}+y^n$$

Since $0<a<1 \Rightarrow \frac{1}{a}>1 \Rightarrow$ we may set $\frac{1}{a}=1+h$, where $h=\frac{1}{a}-1>0$

By definition, we need to show

$\forall \varepsilon>0, \exists n_0$ s.t. $|a^n-0|<\varepsilon \quad \forall n\geq n_0$.

Again the key thing here is to find n_0. We proceed by working backwards:

$$|a^n-0|<\varepsilon \Leftrightarrow a^n<\varepsilon \Leftrightarrow \frac{1}{\varepsilon}<\frac{1}{a^n}$$

$$\Leftrightarrow \frac{1}{\varepsilon}<\left(\frac{1}{a}\right)^n \Leftrightarrow \frac{1}{\varepsilon}<(1+h)^n$$

$$\Leftarrow \frac{1}{\varepsilon}<1+n\cdot h \quad \text{since } (1+h)^n=1+nh+\frac{n(n-1)}{2}h^2+\cdots>1+nh$$

$$\Leftrightarrow n>\frac{1}{h}\left(\frac{1}{\varepsilon}-1\right) \Leftrightarrow n>\left(\frac{1}{a}-1\right)\left(\frac{1}{\varepsilon}-1\right)$$

Since we only need to care about small ε, we may assume $\frac{1}{\varepsilon}-1>0$. Thus picking any

$$n_0 > \left(\frac{1}{a}-1\right)\left(\frac{1}{\varepsilon}-1\right), \text{ e.g. } \left\lceil\left(\frac{1}{a}-1\right)\left(\frac{1}{\varepsilon}-1\right)\right\rceil$$

By the argument above, we then have $\forall n \geq n_0$

$$|a^n - 0| < \varepsilon.$$

To sum up, $\forall \varepsilon>0$, $\exists n_0 = \left\lceil\left(\frac{1}{a}-1\right)\left(\frac{1}{\varepsilon}-1\right)\right\rceil$ s.t.

$$|a^n - 0| < \varepsilon, \quad \forall n \geq n_0$$

$\Rightarrow \lim\limits_{n\to\infty} a^n = 0.$ $\square$

Corollary: $\lim\limits_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$, $\lim\limits_{n\to\infty} e^{-n} = 0$, $\lim\limits_{n\to\infty}\frac{1}{10^n} = 0$, ...

③ Fix $0<a<1$. Show that $\lim\limits_{n\to\infty} n\cdot a^n = 0$

Proof: By definition, we need to show

$\forall \varepsilon>0$, $\exists n_0$ s.t. $|n\cdot a^n - 0| < \varepsilon$ $\forall n \geq n_0$.

We work backwards to find the n_0

$$|n\cdot a^n - 0| < \varepsilon \Leftrightarrow n\cdot a^n < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < \frac{1}{n}\left(\frac{1}{a}\right)^n$$

$$\Leftrightarrow \frac{1}{\varepsilon} < \frac{1}{n}(1+h)^n \qquad \underline{\underline{h = \frac{1}{a}-1}}$$

$$\Leftarrow \frac{1}{\varepsilon} < \frac{1}{n}\,\frac{n(n-1)}{2}\cdot h^2 \qquad \text{since } (1+h)^n = 1+nh+\frac{n(n-1)}{2}h^2+\cdots > \frac{n(n-1)}{2}h^2$$

$$\Leftrightarrow \frac{1}{\varepsilon} < \frac{(n-1)}{2}\cdot h^2$$

$\Leftrightarrow n-1 > \frac{2}{\varepsilon h^2} \Leftrightarrow n > \frac{2}{\varepsilon h^2}+1$

$\Leftrightarrow n > \frac{2a}{\varepsilon(1-a)}+1$

Picking $n_0 = \lceil \frac{2a}{\varepsilon(1-a)}+1 \rceil$, we have

$\forall$ ~~ε~~ $n \geq n_0$, $|na^n - 0| < \varepsilon$.

To sum ~~p~~ up, $\forall \varepsilon > 0$, $\exists n_0 = \lceil \frac{2a}{\varepsilon(1-a)}+1 \rceil$ s.t.

$$|na^n - 0| < \varepsilon \quad \forall n \geq n_0$$

$\Rightarrow \lim_{n\to\infty} na^n = 0$ $\square$

Remark: Comparing examples ② & ③, we may see that the key thing here is that $\left(\frac{1}{a}\right)^n = (1+h)^n$ contains n, n^2 terms. In fact as $n\to\infty$, it contains n^k for any $k \in \mathbb{Z}$. Thus we may guess $\lim_{n\to\infty} n^k a^n = 0$. This is actually true. One of the basic facts in calculus or analysis: ~~P~~ Exponential decaying like a^n $(0<a<1)$ always beats polynomial growth like n^k $(k \in \mathbb{Z}_+)$.

After going through some difficult examples, let's go back to two simple examples.

Example 2.

① Show that constant sequences are convergent & their limits are the single value in the range.

Proof: Let $\{a_n\}_{n\geq 1}$ be a constant sequence. Then $\exists\, c \in \mathbb{R}$ s.t. $a_n \equiv c$. We need to show by definition, i.e. $\forall \varepsilon>0$, $\exists\, n_0$ s.t.

$$|a_n - c| < \varepsilon, \quad \forall n \geq n_0$$

Here since $a_n = c$ for all n, we obtain

$$|a_n - c| = 0 < \varepsilon \quad \text{for all } \varepsilon>0 \text{ \& all } n\geq 1.$$

Thus, $\forall \varepsilon>0$, $\exists\, n_0 = 1$ s.t.

$$|a_n - c| = 0 < \varepsilon \quad \forall n \geq n_0 = 1.$$

$\Rightarrow \lim a_n = c$.

Remark: Geometrically, it's kind of obvious since a_n stay still at c for all $n\geq 1$.

A sequence $\{a_n\}$ is called eventually constant if $\exists\, c \in \mathbb{R}$, $\exists\, n_0 \in \mathbb{Z}_+$ s.t. $a_n = c$ $\forall n \geq n_0$.

② Any eventually constant sequence $\{a_n\}$ is convergent & its limit is the value it takes on for all large n.

Proof: Similarly to ①. $\forall \varepsilon > 0$, $\exists\, n_0$ (the n_0 s.t. $a_n = c$ $\forall n \geq n_0$) s.t.

$$|a_n - c| = 0 < \varepsilon, \ \forall n \geq n_0$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0.$$

Example 3. Divergent sequences.

① Let $a_n = \begin{cases} 1, & n \text{ even} \quad (\text{i.e. } n = 2k,\ k \in \mathbb{Z}_+) \\ -1, & n \text{ odd} \quad (\text{i.e. } n = 2k-1,\ k \in \mathbb{Z}_+). \end{cases}$

Show that it's divergent.

Proof: Theorem 1.2 says that if $\lim_{n\to\infty} a_n = L$, then $\lim_{k\to\infty} a_{n_k} = L$ for any subsequence $\{a_{n_k}\}$ of $\{a_n\}$.

Here we can find two subsequences

$$\{a_{n_k}\} = \{a_{2k}\}_{k\geq 1} = \{1\}_{k\geq 1} \Rightarrow \lim a_{2k} = 1$$

$$\{a_{n_k}\} = \{a_{2k-1}\}_{k\geq 1} = \{-1\}_{k\geq 1} \Rightarrow \lim a_{2k-1} = -1$$

& $1 \neq -1$. Thus $\{a_n\}$ is divergent since it has two convergent subsequences with different limits.