1.3 Operations with Limits

Theorem 1.4 (The Sum Law):
Let $\{a_n\}$ & $\{b_n\}$ be two convergent sequences. Then $\{a_n+b_n\}_{n\geq 1}$ is convergent &

$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n,$$

i.e. the limit of the sum is the sum of the limits.

Proof: We may assume $\lim_{n\to\infty} a_n = A$ & $\lim_{n\to\infty} b_n = B$.
Fix any $\varepsilon>0$. Apply the definition of convergence with $\varepsilon/2$ for both $\{a_n\}$ & $\{b_n\}$, we may find a n_1 s.t. $|a_n - A| < \frac{\varepsilon}{2}$ $\forall n\geq n_1$;
and a n_2 s.t. $|b_n - B| < \frac{\varepsilon}{2}$ $\forall n \geq n_2$.
Set $n_0 = \max\{n_1, n_2\}$. Then $\forall n \geq n_0$, we have $|a_n - A| < \frac{\varepsilon}{2}$ & $|b_n - B| < \frac{\varepsilon}{2}$.
Thus $\forall n\geq n_0$ we have

$$\begin{aligned} |(a_n+b_n)-(A+B)| &= |(a_n-A)+(b_n-B)| \\ &\leq |a_n-A| + |b_n-B| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

i.e. $\forall \varepsilon>0$, $\exists n_0$ s.t. $|(a_n+b_n)-(A+B)|<\varepsilon$ $\forall n\geq n_0$

$\Rightarrow \lim_{n\to\infty}(a_n+b_n) = A+B = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ $\square$

Corollary: Let $\{a_n\}$ be a convergent sequence & $c \in \mathbb{R}$ be a constant. Then

$$\lim_{n\to\infty} (a_n + c) = \lim_{n\to\infty} a_n + c$$

Proof: Taking $b_n \equiv c$. By Thm 1.4

$$\lim_{n\to\infty} (a_n+c) = \lim_{n\to\infty} (a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

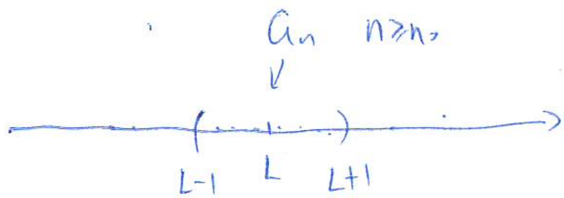
$$= \lim_{n\to\infty} a_n + c$$

Here we use $\lim_{n\to\infty} b_n = c$ □

Definition 1.4. We say a sequence $\{a_n\}$ is bounded if $\exists M>0$ s.t.

$$|a_n| < M \quad \forall n \geq 1.$$

Remark: The boundedness of a sequence & of ~~that~~ a subset $S \subseteq \mathbb{R}$ is slightly different since $\{a_n\}_{n\geq 1}$ is a map & S is a set. However, $\{a_n\}_{n\geq 1}$ is bounded is equivalent to say that the range of $\{a_n\}_{n\geq 1}$ (as a subset of $\mathbb{R}$) is bounded. Indeed, if $\{a_n\}$ is bounded, then the range of $\{a_n\}$ is bounded by (below by $-M$ & above by M). Conversely.



if the range of $\{a_n\}_{n\geq 1}$ is bounded (below by m & above by M), then

$$m \leq a_n \leq M \quad \text{for all } n \geq 1$$

Then by taking $M_0 = \max\{|m|+1, |M|+1\}$, we for sure have $|a_n| < M_0 \quad \forall\, n \geq 1$.

Theorem 1.5: Convergent sequences are bounded.

Proof: Let $\{a_n\}$ be a convergent sequence. ~~Then~~ We may assume $\lim\limits_{n\to\infty} a_n = L$. Then

$\exists\, n_0$ s.t. $\quad |a_n - L| < 1 \quad \forall\, n \geq n_0$.

By triangle inequality,

$$|a_n| - |L| \leq |a_n - L| < 1, \quad \forall n \geq n_0$$

$\Rightarrow \quad |a_n| < |L| + 1 \quad \forall\, n \geq n_0$.

In other words $\{a_n, n \geq n_0\}$ is bounded. But $\{a_n, 1 \leq n \leq n_0 - 1\}$ is bounded since it's a finite set. Now

$$\{a_n, n \geq 1\} = \{a_n;\ 1 \leq n < n_0\} \cup \{a_n;\ n \geq n_0\}$$

is bounded since it's a union of two bounded sets $\Rightarrow$ $\{a_n\}_{n\geq 1}$ is bounded. $\square$

Example: Let $a_n = n$, $\forall n \geq 1$. Then $\{a_n\}$ is divergent.

Proof: $\{a_n\}_{n\geq 1}$ is unbounded since its range is $\mathbb{Z}_+$. Hence it cannot be convergent by Thm 1.5.

Theorem 1.6 (The product Law):

Let $\{a_n\}$ & $\{b_n\}$ be two convergent sequences. Then $\{a_n \cdot b_n\}_{n\geq 1}$ is convergent.

Moreover
$$\lim_{n\to\infty}(a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right)\cdot\left(\lim_{n\to\infty} b_n\right),$$
i.e. the limit of the product is the product of the limits.

Proof: By Thm 1.5, $\{b_n\}_{n\geq 1}$ is bounded.

In particular, $\exists M > 0$ s.t. $|b_n| < M$ $\forall n \geq 1$.

Fix any $\varepsilon > 0$. Assume $\lim_{n\to\infty} a_n = A$ & $\lim_{n\to\infty} b_n = B$.

Then $\lim_{n\to\infty} a_n = A$ implies for $\frac{\varepsilon}{2M}$, $\exists n_1$ s.t.
$$|a_n - A| < \frac{\varepsilon}{2M}, \quad \forall n \geq n_1 \ ;$$
Similarly for $\frac{\varepsilon}{2(|A|+1)}$, $\exists n_2$ s.t.

$$|b_n - B| < \frac{\varepsilon}{2(|A|+1)} \qquad \forall n \geq n_2$$

Set $n_0 = \max\{n_1, n_2\}$. Then for $n \geq n_0$, we have

$$\begin{cases} |a_n - A| < \frac{\varepsilon}{2M} \\ |b_n - B| < \frac{\varepsilon}{2(|A|+1)} \end{cases}.$$

Then $\forall n \geq n_0$

$$\begin{aligned}
|(a_n \cdot b_n) - A \cdot B| &= |a_n \cdot b_n - Ab_n + Ab_n - AB| \\
&\leq |a_n b_n - Ab_n| + |Ab_n - AB| \\
&= |b_n| \cdot |a_n - A| + |A| \cdot |b_n - B| \\
&< M \cdot \frac{\varepsilon}{2M} + |A| \cdot \frac{\varepsilon}{2(|A|+1)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

To sum up, $\forall \varepsilon > 0$, $\exists n_0$ s.t.

$$|a_n \cdot b_n - AB| < \varepsilon \qquad \forall n \geq n_0$$

$\Rightarrow \lim_{n \to \infty} (a_n \cdot b_n) = A \cdot B = \left(\lim_{n \to \infty} a_n\right) \cdot \left(\lim_{n \to \infty} b_n\right)$ □

Corollary. Let $\{a_n\}$ be a convergent sequence. Then $\lim_{n \to \infty} (c a_n) = c \lim_{n \to \infty} a_n$ for any constant $c \in \mathbb{R}$.

Theorem 1.7: Let $\{b_n\}$ be a convergent sequence with a nonzero limit. In other words, $\lim_{n\to\infty} b_n = B$ & $B \neq 0$. Then

$\exists n^*$ s.t. $|b_n| > \frac{|B|}{2}$ $\forall n \geq n.$

Proof: Taking $\varepsilon = \frac{|B|}{2} > 0$, then

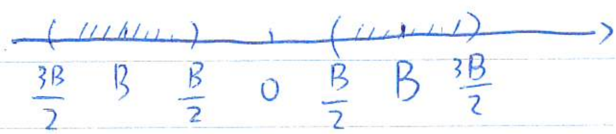
$\lim_{n\to\infty} b_n = B$ implies that $\exists n^*$ s.t.

$$|b_n - B| < \frac{|B|}{2}, \quad \forall n \geq n^*$$

By triangle inequality,

$$|B| - |b_n| < \frac{|B|}{2}, \quad \forall n \geq n^*$$

$$\Rightarrow |b_n| > |B| - \frac{|B|}{2} = \frac{|B|}{2}, \quad \forall n \geq n^* \qquad \square$$



Theorem 1.8 (The Quotient Law)

Let $\{a_n\}$ & $\{b_n\}$ be two convergent sequences & $\lim_{n\to\infty} b_n \neq 0$. Then

$\left\{\frac{a_n}{b_n}\right\}_{n\geq 1}$ converges & $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

i.e. the limit of the quotient is the gotient of the limits.

Proof: Since $\lim_{n\to\infty} b_n = B$ with $B \neq 0$,
we may find a $n^* \in \mathbb{Z}_+$ s.t.

$$|b_n| > \frac{|B|}{2} \qquad \forall\, n \geq n_*.$$

Fix any $\varepsilon > 0$. (Again we assume $\lim a_n = A$)

Then for $\frac{\varepsilon |B|}{4}$, we can find a n_1 s.t.

$$|a_n - A| < \frac{\varepsilon |B|}{4}, \quad \forall\, n \geq n_1;$$

for $\frac{|B|^2}{4(|A|+1)} \cdot \varepsilon$, we can find a n_2 s.t.

$$|b_n - B| < \frac{|B|^2 \cdot \varepsilon}{4(|A|+1)}, \quad \forall\, n \geq n_2$$

Set $n_0 = \max\{n^*, n_1, n_2\}$. Then $\forall\, n \geq n_0$
we have

$$\begin{cases} |b_n| > \frac{|B|}{2} \\ |a_n - A| < \frac{\varepsilon\, |B|}{4} \\ |b_n - B| < \frac{\varepsilon\, |B|^2}{4(|A|+1)} \end{cases}$$

Then we have $\forall\, n \geq n_0$ that:

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$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| = \left|\frac{a_n B - A b_n}{b_n \cdot B}\right|$$

$$= \frac{|a_n B - AB + AB - A b_n|}{|b_n| \cdot |B|}$$

$$< \frac{|B|\,|a_n - A| + |A| \cdot |B - b_n|}{\frac{|B|}{2} \cdot |B|}$$

$$< \frac{2}{|B|^2}\left(|B| \cdot \frac{\varepsilon |B|}{4} + |A| \cdot \frac{\varepsilon |B|^2}{4(|A|+1)}\right)$$

$$< \frac{2|B|^2}{|B|^2}\left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right)$$

$$= \varepsilon.$$

To sum up, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{Z}_+$ s.t.

$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| < \varepsilon \quad \forall n \geq n_0$

$\Rightarrow \lim\limits_{n \to \infty} \frac{a_n}{b_n} = \frac{A}{B} = \frac{\lim a_n}{\lim b_n}$ $\square$

Corollary: Let $\{a_n\}$ be a convergent sequence & $\lim a_n \neq 0$. Then $\lim\limits_{n \to \infty} \left(\frac{c}{a_n}\right) = c / \lim\limits_{n \to \infty} a_n$

* From now on, for a sequence $\{a_n\}$, if we say something holds true for all n sufficient larg, then it means $\exists\, n^* \in \mathbb{Z}_+$ s.t. it holds true for all $n \geq n^*$

Theorem 1.9 (The Squeeze Theorem).

Suppose $\{a_n\}$, $\{b_n\}$, & $\{c_n\}$ are sequenies s.t. $a_n \leq b_n \leq c_n$ for all n sufficient large. More $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = A$. Then

$$\lim_{n\to\infty} b_n = A.$$

Proof: We may fix a n^* s.t.

$$a_n \leq b_n \leq c_n \quad \forall\, n \geq n^*. \qquad ①$$

~~By~~ Fix any $\varepsilon > 0$. By definition,

$\exists\, n_1$ s.t. $|a_n - A| < \varepsilon/3 \quad \forall\, n \geq n_1$; ②

$\exists\, n_2$ s.t. $|c_n - A| < \varepsilon/3 \quad \forall\, n \geq n_2$ ③

Set $n_0 = \max\{n^*, n_1, n_2\}$. Then we have ①, ②, & ③ $\forall\, n \geq n_0$. Then $\forall\, n \geq n_0$

it holds that

$$|b_n - A| = |b_n - c_n + c_n - A|$$

$$\leq c_n - b_n + |c_n - A| \qquad c_n - b_n = |b_n - c_n| \text{ as } c_n \geq b_n \ \forall n \geq n_*$$

$$\leq c_n - a_n + |c_n - A| \qquad b_n \geq a_n \ \forall n \geq n_*$$

$$= |c_n - a_n| + |c_n - A|$$

$$\leq |c_n - A| + |a_n - A| + |c_n - A|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \qquad n \geq n_1, n \geq n_2$$

$$= \varepsilon$$

To sum up, $\forall \varepsilon > 0, \exists n_0$ s.t. $|b_n - A| < \varepsilon \ \forall n \geq n_0$

$\Rightarrow \lim_{n \to \infty} b_n = A$ $\square$

Theorem 1.10 Let $\{a_n\}$ & $\{b_n\}$ be two convergent sequences. Suppose $a_n \leq b_n$ for all n sufficiently large. Then $\lim a_n \leq \lim b_n$.

Proof: Let $A = \lim a_n$ & $B = \lim b_n$.

Fix n_* s.t. $a_n \leq b_n \ \forall n \geq n_*$.

Find n_1 s.t. $|a_n - A| < \frac{\varepsilon}{2} \ \forall n \geq n_1$;

n_2 s.t. $|b_n - B| < \frac{\varepsilon}{2} \ \forall n \geq n_2$

Set $n_0 = \max\{n_*, n_1, n_2\}$. Then $\forall n \geq n_0$, we have

$$a_n \leq b_n \quad ①$$

$$A - a_n \leq |A - a_n| < \frac{\varepsilon}{2} \Rightarrow A < a_n + \frac{\varepsilon}{2} \quad ②$$

$$b_n - B \leq |b_n - B| < \frac{\varepsilon}{2} \Rightarrow b_n < B + \frac{\varepsilon}{2} \quad ③$$

$$\Rightarrow \forall n \geq n_0, \quad A \overset{②}{<} a_n + \frac{\varepsilon}{2} \overset{①}{\leq} b_n + \frac{\varepsilon}{2} \overset{③}{<} B + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = B + \varepsilon$$

$\Rightarrow A < B + \varepsilon, \ \forall \varepsilon > 0.$

$\Rightarrow A - B < \varepsilon, \ \forall \varepsilon > 0.$ This is only possible when $A - B \leq 0 \Rightarrow A \leq B \Rightarrow \lim a_n \leq \lim b_n$ □

Example 1. Compute $\lim\limits_{n\to\infty} \frac{n^4 + n^3 + 2n^2}{2n^4 + n^2 + 1}$

Solution: $\lim\limits_{n\to\infty} \frac{n^4+n^3+2n^2}{2n^4+n^2+1} = \lim\limits_{n\to\infty} \frac{(n^4+n^3+2n^2)/n^4}{(2n^4+n^2+1)/n^4}$

$$= \lim_{n\to\infty} \frac{1 + \frac{1}{n} + \frac{2}{n^2}}{2 + \frac{1}{n^2} + \frac{1}{n^4}}$$

Note $\lim\left(2 + \frac{1}{n^2} + \frac{1}{n^4}\right) \overset{S.L}{=} 2 + \lim \frac{1}{n^2} + \lim \frac{1}{n^4} = 2 \neq 0$

By Q.L. the limit above is

$$\overset{Q.L.}{=} \frac{\lim\left(1 + \frac{1}{n} + \frac{2}{n^2}\right)}{\lim\left(2 + \frac{1}{n^2} + \frac{1}{n^4}\right)} \overset{S.L.}{=} \frac{1 + \lim\frac{1}{n} + \overbrace{2\lim\frac{1}{n^2}}^{P.L.}}{2} = \frac{1}{2} \quad □$$

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Example 2. Show that $\left\{ a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}} \right\}$ is convergent & compute its limit.

Solution. Clearly for each $1 \le k \le n$, it holds that $\frac{1}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+k}} \le \frac{1}{\sqrt{n^2+1}}$. Thus

$$\underbrace{\frac{1}{\sqrt{n^2+n}} + \dots + \frac{1}{\sqrt{n^2+n}}}_{n \text{ ~~times~~ terms}} \le a_n \le \underbrace{\frac{1}{\sqrt{n^2+1}} + \dots + \frac{1}{\sqrt{n^2+1}}}_{n \text{ terms}}$$

$$\Rightarrow \frac{n}{\sqrt{n^2+n}} \le a_n \le \frac{n}{\sqrt{n^2+1}}$$

But $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} \overset{Q.L.}{=\!=\!=} \frac{1}{\lim \sqrt{1+\frac{1}{n}}} = \frac{1}{1} = 1$

$1 \le \sqrt{1+\frac{1}{n}} \le 1+\frac{1}{n}$ & $\lim (1+\frac{1}{n}) = 1$
By Thm 1.9, $\lim \sqrt{1+\frac{1}{n}} = 1 \ne 0$

Similarly $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} \overset{Q.L.}{=\!=\!=} \frac{1}{\lim \sqrt{1+\frac{1}{n^2}}} = \frac{1}{1} = 1$

$1 \le \sqrt{1+\frac{1}{n^2}} \le 1+\frac{1}{n^2}$ & $\lim (1+\frac{1}{n^2}) = 1$
By Thm 1.9, $\lim \sqrt{1+\frac{1}{n^2}} = 1$

By Thm 1.9 again, a_n converges & $\lim_{n\to\infty} a_n = 1$ $\square$