# 1.4 Monotone Sequences

Definition 1.5: Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers.

① $\{a_n\}$ is said to be monotone increasing if

$$a_n \leq a_{n+1} \quad \forall\, n \geq 1.$$

② $\{a_n\}$ is said to be monotone decreasing if

$$a_n \geq a_{n+1} \quad \forall\, n \geq 1.$$

③ A monotone sequence is a sequence that is either monotone increasing or monotone decreasing.

Remark: If the "inequality sign" ($\leq$ or $\geq$) becomes "strictly inequality sign" ($<$ or $>$), then we simply put "strictly" in front of monotone. For instance, $\{a_n\}$ is strictly monotone increasing

$$\text{if} \quad a_n < a_{n+1} \quad \forall\, n \geq 1.$$

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First answer to the question: what is $\mathbb{R}$?

$\mathbb{R}$ is the set containing $\mathbb{Q}$ with the least-upper-bound property. Specifically.

For each $S \subseteq \mathbb{R}$ that is bounded above, (i.e. $\exists M \in \mathbb{R}$ s.t. $x \leq M \;\forall x \in S$) there is a least-upper-bound $\alpha$ in the sense that: (i) $\alpha$ is an upper-bound of $S$ (i.e. $x \leq \alpha \;\forall x \in S$); (ii) No number smaller than $\alpha$ is an upper-bound of $S$, i.e. $\forall \varepsilon>0$, $\exists x_0 \in S$ s.t.

$$x_0 > \alpha-\varepsilon$$

denoted $\alpha = \sup S$

(No matter how small $\varepsilon$ is, there is some number in $S$ that is strictly bigger than $\alpha-\varepsilon$). Such an $\alpha$ is called the least upper bound or supremum of $S$.

In some sense, the least-upper-bound property of $\mathbb{R}$ is not something that can be proved. It is the property defining what is $\mathbb{R}$.

Theorem 1.11: If a monotone increasing sequence is bounded above, then it's convergent.

Proof: Let $\{a_n\}$ be a monotone increasing sequence that is bounded above. So the range $\{a_n, n\geq 1\}$ is bound above. By the least-upper-bound property of $\mathbb{R}$, we may set

$$A = \sup\{a_n, n\geq 1\}.$$

i.e. $A$ is the least-upper-bound ~~p~~ or supremum of $\{a_n, n\geq 1\}$. Then,

(i) $a_n \leq A$, $\forall n \geq 1$

(ii) $\forall \varepsilon>0$, $\exists n_0$ s.t. $a_{n_0} > A-\varepsilon$.

But $\{a_n\}$ is monotone increasing, (ii) implies that $\forall n\geq n_0$, $a_n \geq a_{n_0} > A-\varepsilon$. Thus we may restate (ii) as:

$x \in (A-\varepsilon, A+\varepsilon) \Leftrightarrow |x-A| < \varepsilon$

$x \in (A-\varepsilon, A] \Rightarrow x \in (A-\varepsilon, A+\varepsilon) \Rightarrow |x-A| < \varepsilon$

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(ii)' $\forall \varepsilon > 0, \exists n_0$ s.t. $a_n > A - \varepsilon \quad \forall n \geq n_0$

Combining (i) & (ii)', we obtain:

$\forall \varepsilon > 0, \exists n_0$ s.t.


$$A - \varepsilon < a_n \leq A \qquad \forall n \geq n_0$$

$$\Rightarrow |a_n - A| < \varepsilon, \quad \forall n \geq n_0$$

$\Rightarrow \lim_{n\to\infty} a_n = A.$ $\square$

Corollary: If $\{a_n\}$ is monotone decreasing & bound below, then it's convergent.

Proof: We claim $\{-a_n\}$ is monotone increasing & bounded above. Indeed,

(i) $a_n \geq a_{n+1} \;\forall n \geq 1 \Rightarrow -a_n \leq -a_{n+1} \quad \forall n \geq 1$

(ii) $a_n \geq m, \forall n \geq 1 \Rightarrow -a_n \leq -m \quad \forall n \geq 1$

as desired. Thus we can apply Thm 1.11 to $\{-a_n\}$ & obtain that

$$\lim_{n\to\infty} (-a_n) \text{ exists.}$$

By product law:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} -1 \cdot (-a_n) = -\lim_{n\to\infty} (-a_n)$$

In particular, $\{a_n\}$ converges. $\square$

Theorem 1.12: The sequence

$$a_n = (1+\tfrac{1}{n})^n$$

is monotone increasing & $2 < a_n < 3$, $\forall n \geq 1$.
In particular, it's convergent.

Proof: By binomial expansion,

$$a_n = (1+\tfrac{1}{n})^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \cdot (\tfrac{1}{n})^k = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}$$

For each $0 \leq k \leq n$, $\binom{n}{k} \frac{1}{n^k} = \dfrac{n(n-1)\cdots(n-k+1)}{k! \cdot n^k}$

$$= \frac{1}{k!} \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n} = \frac{1}{k!}(1-\tfrac{1}{n})(1-\tfrac{2}{n})\cdots(1-\tfrac{k-1}{n})$$

Thus $a_n = 1+1+\frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \cdots +$

$$\frac{1}{n!}(1-\tfrac{1}{n})\cdots(1-\tfrac{n-1}{n}). \qquad (*)$$

It implies that: $a_{n+1} = 1+1+\frac{1}{2!}(1-\frac{1}{n+1}) + \frac{1}{3!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})$
$+ \cdots + \frac{1}{n!}(1-\frac{1}{n+1})\cdots(1-\frac{n-1}{n+1}) + \frac{1}{(n+1)!}(1-\frac{1}{n+1})\cdots(1-\frac{n}{n+1})$.

Clearly, $\frac{1}{k!}(1-\frac{1}{n})\cdots(1-\frac{k-1}{n}) < \frac{1}{k!}(1-\frac{1}{n+1})\cdots(1-\frac{k-1}{n+1})$, $\forall\, 2 \leq k \leq n$

$\Rightarrow$ The kth term of $a_{n+1}$ > The kth term of $a_n$.

$\Rightarrow a_{n+1} > a_n \quad \forall n \geq 1$

$\Rightarrow \{a_n\}$ is strictly monotone increasing.

Also, by the formula $(*)$, we can see that

$$2 < a_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

Clearly, $n! = n(n-1)\cdots 2\cdot 1 \geq \underbrace{2\cdot 2 \cdots 2}_{n-1} = 2^{n-1} \quad \forall n \geq 1.$

$\Rightarrow \frac{1}{n!} \leq \frac{1}{2^{n-1}}, \quad \forall n \geq 1.$

$\Rightarrow \quad 2 < a_n < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$

But $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = \dfrac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 2 - (\frac{1}{2})^{n-1}$

$\Rightarrow 2 < a_n < 1 + 2 - (\frac{1}{2})^{n-1} = 3 - (\frac{1}{2})^{n-1} < 3 \quad \forall n \geq 1.$

Since $\{a_n\}$ is monotone increasing & bounded above (by 3), it's convergent by Thm 1.11 $\square$

We denote $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(1+\frac{1}{n})^n$ by $e$, i.e.

$$e = \lim_{n\to\infty}(1+\frac{1}{n})^n.$$

Example 1: Compute $\lim_{n\to\infty}(1+\frac{1}{2n})^n$ & $\lim_{n\to\infty}(1+\frac{1}{n})^{\frac{n}{2}}$

Solution. ① $\{b_n = (1+\frac{1}{2n})^{2n}\}$ is a subsequence of $\{a_n = (1+\frac{1}{n})^n\}$. So it's monotone increasing & bounded above. Thus $\{\sqrt{b_n} = (1+\frac{1}{2n})^n\}$ is monotone increasing & bounded above. By Thm 1.11 $\lim_{n\to\infty}(1+\frac{1}{2n})^n$ exists

By product law $\lim_{n\to\infty}(1+\frac{1}{2n})^n \cdot \lim_{n\to\infty}(1+\frac{1}{2n})^n$

$= \lim_{n\to\infty}\left[(1+\frac{1}{2n})^n \cdot (1+\frac{1}{2n})^n\right] = \lim_{n\to\infty}(1+\frac{1}{2n})^{2n} = \lim_{n\to\infty} b_n = e$

since $\{b_n\}$ is a subsequence of $\{a_n\}$

$\Rightarrow \left(\lim_{n\to\infty}(1+\frac{1}{2n})^n\right)^2 = e \Rightarrow \lim_{n\to\infty}(1+\frac{1}{2n})^n = \sqrt{e}$

② Since $\{a_n = (1+\frac{1}{n})^n\}$ is monotone increasing & bounded above $\Rightarrow \{\sqrt{a_n} = (1+\frac{1}{n})^{\frac{n}{2}}\}$ is monotone increasing & bounded above $\Rightarrow \lim_{n\to\infty}(1+\frac{1}{n})^{\frac{n}{2}}$ exists

$\Rightarrow \lim_{n\to\infty}(1+\frac{1}{n})^{\frac{n}{2}} \lim_{n\to\infty}(1+\frac{1}{n})^{\frac{n}{2}} = \lim_{n\to\infty}(1+\frac{1}{n})^{\frac{n}{2}} \cdot (1+\frac{1}{n})^{\frac{n}{2}}$

$= \lim_{n\to\infty}(1+\frac{1}{n})^n = e$

$\Rightarrow \left(\lim_{n\to\infty}(1+\frac{1}{n})^{\frac{n}{2}}\right)^2 = e \Rightarrow \lim_{n\to\infty}(1+\frac{1}{n})^{\frac{n}{2}} = \sqrt{e}$ □

Definition 1.6: We say $\lim_{n\to\infty} a_n = \infty$ if $\forall N \in \mathbb{Z}_+, \exists n_0$ s.t. $a_n \geq N$, $\forall n \geq n_0$. (Note such $\{a_n\}$ is divergent)

Theorem 1.13: Let $\{a_n\}$ be monotone increasing & unbounded. Then $\lim_{n\to\infty} a_n = \infty$.

Proof: Since $\{a_n\}$ is unbounded, $\forall N \in \mathbb{Z}_+$, $\exists n_0$ s.t. $a_{n_0} \geq N \Rightarrow \forall n \geq n_0$, $a_n \geq a_{n_0} \geq N$

$\Rightarrow \lim_{n\to\infty} a_n = +\infty$.

* Similarly, one can show that if $\{a_n\}$ is monotone decreasing & unbounded, then $\lim_{n\to\infty} a_n = -\infty$