1.5 Bolzano-Weierstrass Theorem

Defition 1.6: Let $S \subseteq \mathbb{R}$ be a subset. We say $\alpha \in \mathbb{R}$ is a limit point of S if:

$$\forall \varepsilon > 0, \ \exists \ a \in S \setminus \{\alpha\} \quad s.t.$$

$$|a - \alpha| < \varepsilon.$$

We may also call such an α a cluster point or a point of accumulation of S.

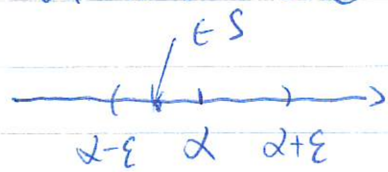
Remark: Note that the notion of limit pt describe the relation between a subset S of $\mathbb{R}$ & a point $\alpha \in \mathbb{R}$. Here α might or might not belong to S.

The following theorem tells us the relation between a limit point of a set & the limit of a convergent sequence.

Theorem 1.14. If α is a limit point of S, then there exists a sequence $\{a_n\}$ of mutually distinct points that belong to S s.t.

$$\lim_{n \to \infty} a_n = \alpha.$$

Geometrically it's not so hard to visualize it. The definition 1.6 basically says that no matter how small $\varepsilon > 0$ is, the open interval centered at α always contains at least one point of $S \setminus \{\alpha\}$.



In particular, for each n

if we set $\varepsilon = \frac{1}{n}$, then we will find a point $a_n \in S\setminus\{\alpha\}$ s.t. $|a_n - \alpha| < \frac{1}{n}$. We can then show $\lim a_n = \alpha$.

But we need to make sure that a_n's are mutually different, i.e. $a_n \neq a_k \ \forall\, n \neq k$

Proof: By definition 1.6, for $\varepsilon_1 = 1$, we can find a ~~$a_1 \neq \alpha$ s.t.~~ $a_1 \in S \setminus \{\alpha\}$ s.t.

$$|a_1 - \alpha| < 1$$

Then we set $\varepsilon_2 = \min\{\frac{1}{2}, |a_1 - \alpha|\} > 0$. For ε_2, we can find a $a_2 \in S \setminus \{\alpha\}$ s.t.

$$|a_2 - \alpha| < \varepsilon_2$$

Note $|a_2 - \alpha| < \varepsilon_2 \leq \frac{1}{2}$ & $|a_2 - \alpha| < \varepsilon_2 \leq |a_1 - \alpha|$

which implies that $a_2 \neq a_1$.

Proceeds like this. We may then obtain for each $n \in \mathbb{Z}^+$ & each

$$\varepsilon_n = \min\{\tfrac{1}{n}, |a_1 - \alpha|, \cdots, |a_{n-1} - \alpha|\} > 0$$

a point $a_n \in S \setminus \{\alpha\}$ s.t.

$$|a_n - \alpha| < \varepsilon_n$$

Note that $|a_n - \alpha| < \varepsilon_n \leq \frac{1}{n}$ &

$|a_n - \alpha| < \varepsilon_n \leq |a_k - \alpha| \quad \forall\, k = 1, \cdots, n-1$ which implies that $a_n \neq a_1, a_n \neq a_2, \cdots, a_n \neq a_{n-1}$.

Thus we obtain a sequence of mutually different points $\{a_n\}_{n \geq 1}$ in $S \setminus \{\alpha\}$ s.t.

$$|a_n - \alpha| < \frac{1}{n}$$

$$\Rightarrow \quad -\frac{1}{n} < a_n - \alpha < \frac{1}{n}$$

But $\lim\limits_{n \to \infty} (-\frac{1}{n}) = \lim\limits_{n \to \infty} (\frac{1}{n}) = 0$. By squeeze thm

$$\lim_{n \to \infty} (a_n - \alpha) = 0$$

By sum & product laws: $a_n = a_n - \alpha + \alpha$ conv.

$\boxed{\cancel{0 = \lim\limits_{n \to \infty} (a_n - \alpha) = \lim\limits_{n \to \infty}}}$ & its limit is:

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (a_n - \alpha + \alpha) = \lim_{n \to \infty} (a_n - \alpha) + \alpha$$

$$= \alpha.$$

□

Remark: Obviously, the converse statement of Thm 1.14 is true as well. Indeed, if $\exists \{a_n\}_{n\geq 1}$ s.t. $a_n \neq a_k$ $\forall n, k \in \mathbb{Z}_+$ & $a_n \in S \setminus \{\alpha\}$

& $\lim\limits_{n\to\infty} a_n = \alpha$.

Then $\forall \varepsilon > 0$, $\exists n_0$ s.t. $|a_n - \alpha| < \varepsilon$, $\forall n \geq n_0$.

In particular, there is a $a_n \in S \setminus \{\alpha\}$ s.t.

$$|a_n - \alpha| < \varepsilon.$$

$\Rightarrow$ ~~a~~ α is a limit pt of S. Thus Thm 1.14 is actually "if & only if"

Corollary: A finit set S cannot have limit pt.

Proof: By Thm 1.14, if S has a limit pt α, then ~~a~~ S contains a sequence of mutually distinct pts $\{a\}_{n\geq 1}$. Clearly, card $\{a_n, n\geq 1\}$ is infinite. But $\{a_n, n\geq 1\} \subseteq S$ which implies that Card(S) is infinite. In other words, we showed that if a set S has a limit pt, then its cardinality must be infinite $\Rightarrow$ finite set cannot have limit pts $\square$

Definition 1.7: We define S' to be the set of limit points of S, i.e. $\alpha \in S'$ if & only if α is a limit point of S. We define $\bar{S} := S \cup S'$ to be the closure of S. If $S' \subseteq S$, or equivalently $\bar{S} = S$, then we say S is a closed set.

Example 1

① A finite set S is closed.

Solution: Let S be a finite set. Then by the corollary of Thm 1.14, $S' = \emptyset \Rightarrow S' \subset S$ $\Rightarrow S$ is closed.

② A closed interval $I = [a, b]$ is closed.

Solution: We need to show $I' \subset I$. Equivalently, we show $\forall x \notin I$, $x \notin I'$ (i.e. $I^c \subset (I')^c$). If $x \notin I$, then either $x < a$ or $x > b$. Say $x < a$. Then we set $\varepsilon = a - x > 0$, & consider the interval $(x-\varepsilon, x+\varepsilon)$. For all $y \in (x-\varepsilon, x+\varepsilon)$, we have

$y < x + \varepsilon = x + (a - x) = a \Rightarrow y \notin I = [a, b]$

$\Rightarrow (x - \varepsilon, x + \varepsilon) \cap I = \phi$. In other words, for any $x < a$, we can find an $\varepsilon > 0$ s.t. there is no point z in I with $|z - x| < \varepsilon$.

$\Rightarrow x \notin I'$. Similarly, we can show if $\cancel{y}\ x > b$, then $\cancel{y}\ x \notin I'$. $\Rightarrow x \notin I'$ if $x \notin I$

$\Rightarrow I' \subset I$. $\square$

③. An open interval $I = (a, b)$ is not closed.

Solution: In fact, we can show $a \in I'$. But clearly $a \notin I$. Thus $I' \not\subset I \Rightarrow I$ is not closed.

$\forall \varepsilon > 0$, it clearly holds that

$$(a - \varepsilon, a + \varepsilon) \cap (a, b) = \begin{cases} (a, a + \varepsilon), & \text{if } a + \varepsilon \le b \\ (a, b) & \text{if } a + \varepsilon > b \end{cases}$$

$\neq \phi$. $\Rightarrow \exists\, x \in (a - \varepsilon, a + \varepsilon) \cap I$

$\Rightarrow |x - a| < \varepsilon$, $x \in I$ (and $x \neq a$ since $a \notin I$).

$\Rightarrow a$ is a limit point of I

& $a \notin I \Rightarrow I$ is not closed.

④ $S = \{\frac{1}{n}\,;\ n \in \mathbb{Z}_+\}$ is not closed. (In fact $\bar{S} = \{0, \frac{1}{n}\,;\ n \in \mathbb{Z}_+\}$)

Solution: clearly $\{a_n = \frac{1}{n}\}_{n \geq 1}$ is a sequence of mutually distint points of S &

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0$$

By the remark following Thm 1.14. $0 \in S'$

But $0 \notin S$. Thus S is not closed.

Definition 1.8: Let $\{I_n\}_{n \geq 1}$ be a sequence of closed intervals s.t.

$$I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots$$

Let λ_n be the length of I_n. We say $\{I_n\}_{n \geq 1}$ is a nest if in addition $\lim_{n \to \infty} \lambda_n = 0$.

Theorem 1.15. Let $\{I_n\}_{n \geq 1}$ be a nest. Then there is a $\xi \in \mathbb{R}$ s.t. $\{\xi\} = \bigcap_{n=1}^{\infty} I_n$, i.e. the intersection of I_n, $n \geq 1$ is a single point set.

Proof: We write $I_n = [a_n, b_n]$. $n \geq 1$.

Since $I_n = [a_n, b_n] \supseteq [a_{n+1}, b_{n+1}] = I_n$

$\Rightarrow a_n \le a_{n+1} \le b_{n+1} \le b_n \quad \forall n \ge 1.$

In particular, $\{a_n\}_{n\ge 1}$ is a monotone increasing sequence that is bounded above (e.g. by b_1, or any b_n). By Theorem 1.11 $\{a_n\}$ is convergent, we may set

$$\lim_{n\to\infty} a_n = \xi.$$

Similarly, $\{b_n\}$ is a monotone decreasing sequence that is bounded below. Thus $\exists \eta \in \mathbb{R}$ s.t. $\lim_{n\to\infty} b_n = \eta$.

By Theorem 1.10, $a_n \le b_n \quad \forall n \ge 1$

$$\Rightarrow \xi \le \eta$$

Recall from the proof of Thm 1.11, ξ is the $\sup\{a_n\} \Rightarrow a_n \le \xi \quad \forall n \ge 1$.

Similarly, $\eta = \inf\{b_n\}$, the greatest-lower-bound or the infimum of $\{b_n\}$. $\Rightarrow \eta \le b_n \quad \forall n \ge 1$

$\Rightarrow a_n \le \xi \le \eta \le b_n \quad \forall n \ge 1$

$\Rightarrow \xi \in I_n \ \forall n \ge 1 \Rightarrow \xi \in \bigcap_{n\ge 1} I_n.$

Next, we need to show ξ is the only point in $\bigcap_{n=1}^{\infty} I_n$. Suppose $\boxed{\text{for the sake of contradiction}}$, there is $\xi' \in \bigcap_{n=1}^{\infty} I_n$, $\boxed{\xi' \neq \xi}$ Then

$$a_n \le \xi' \le b_n \quad \forall\, n \ge 1$$

$$\& \quad a_n \le \xi \le b_n \quad \forall\, n \ge 1$$

$$\Rightarrow 0 < |\xi - \xi'| \le |b_n - a_n| = \lambda_n. \quad \forall\, n \ge 1$$

But $\lim_{n \to \infty} \lambda_n = 0$. By Squeeze Thm,

$$|\xi - \xi'| = 0$$

$\Rightarrow \xi' = \xi$. $\Rightarrow$ $\bigcap_{n=1}^{\infty} I_n$ contains only ξ, i.e.

$$\bigcap_{n=1}^{\infty} I_n = \{\xi\} \qquad \square$$

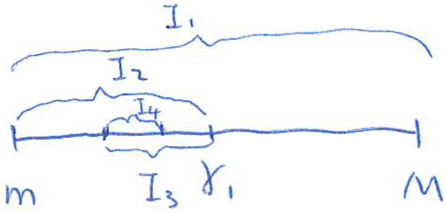
Theorem 1.16 (Bolzano–Weierstrauss Thm for sets)

Every bounded, infinite set of real numbers has at least one limit point.

Proof: Let S be such a bound & infinite set. Then $\exists\ m \le M$ s.t.

$$m \le x \le M, \quad \forall x \in S, \text{ or } S \subseteq [m, M]$$

We denote $[m, M]$ by I_1.



Denote by γ_1 the midpoint $\frac{m+M}{2}$ of I_1 & consider two closed intervals

$[m, \gamma_1]$ & $[\gamma_1, M]$

Then at least one of them contains an infinite number of points in S.

Denote that interval by I_2.

Then we can repeat the same process we did on I_1 on I_2 & obtain a ~~a half~~ closed interval I_3 that contains infinite number of points of S. & I_3 is basically half of I_2.

Proceed by induction, we can then obtain a sequence of closed intervals $\{I_n\}_{n\geq 1}$ s.t.

① $I_n \supset I_{n+1}$, $\forall n \geq 1$

② $|I_n| = \frac{M-m}{2^{n-1}} \Rightarrow \lim |I_n| = \lim \frac{M-m}{2^{n-1}} = 2(M-m)\lim \frac{1}{2^n} = 0$

③ I_n contains an infinite number of pts in S

$\Rightarrow \{I_n\}_{n\geq 1}$ is a nest $\Rightarrow \exists \xi$ s.t. $\{\xi\} = \bigcap_{n=1}^{\infty} I_n$. Now $\forall \varepsilon > 0$, $\exists n$ s.t. $\frac{M-m}{2^{n-1}} < \varepsilon$

Since $\xi \in I_n$, $|I_n| = \frac{M-m}{2^{n-1}} < \varepsilon$ & I_n contains infinite number of points in S, we may in particular pick a point $x \in I_n \ni x \in S,\ x \neq \xi$

Since $x, \xi \in I_n \Rightarrow |x-\xi| < |I_n| < \varepsilon$.

To summarize, $\forall \varepsilon > 0,\ \exists\, x \in S \setminus \{\xi\}$ s.t.

$$|x-\xi| < \varepsilon$$

$\Rightarrow \xi \in S'$ $\square$

Theorem (1.17): (Balzano-Weierstrass Thm for sequences).

Every bounded sequence of real numbers has at least one convergent subsequence.

proof: Let $\{a_n\}$ be a bounded sequence. We divide it into two different cases.

Case I: The range of $\{a_n\}$ is a finite set.

Then at least one of points in the range $\{a_n, n \geq 1\}$ must occur infinitely number of times in the sequence $\{a_n\}$. Otherwise, the sequence is finite since each pt in its range occurs in the sequence finite number of times.

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Specifically, there is a $\xi \in \{a_n, n\geq 1\}$ s.t.
$\exists$ $n_1 < n_2 < \cdots < n_k < \cdots$ of $\mathbb{Z}_+$ s.t.

$$a_{n_k} = \xi \quad \forall k \geq 1$$

$\Rightarrow \lim_{k\to\infty} a_{n_k} = \xi$ since the subsequence $\{a_{n_k}\}$ is a constant sequence.

Case II: The range $S = \{a_n ; n\geq 1\}$ is an infinite set.
Then by Thm 1.16, S has a limit pt ξ.
By definition, there is a $a_{n_1} \in S\setminus\{\xi\}$ s.t.

$$|a_{n_1} - \xi| < 1$$

Set $\underset{>0}{\varepsilon_2 = \min}\{\frac{1}{2}, |a_1-\xi|, \ldots, |a_{n_1}-\xi|, a_n \neq \xi \; 1\leq n\leq n_1\}$

Then we may find n_2 with $a_{n_2} \in S\setminus\{\xi\}$ s.t.

$$|a_{n_2} - \xi| < \varepsilon_2$$

Since $|a_{n_2} - \xi| \neq |a_n - \xi|$, $\forall\, 1\leq n\leq n_1 \Rightarrow n_2 > n_1$.
Proceed by induction, we find a sequence $\{a_{n_k}\}_{k\geq 1}$ with $n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots$

s.t. $|a_{n_k} - \xi| < \varepsilon_k \leq \frac{1}{k} \quad \forall k \geq 1.$

$\Rightarrow \{a_{n_k}\}$ is a subsequence of $\{a_n\}$ &

$$-\frac{1}{k} < a_{n_k} - \xi < \frac{1}{k} \Rightarrow \lim_{n\to\infty} a_{n_k} = \xi$$

$\square$