MATH 150A Intermediate Analysis

Chapter 1: Sequences & Limits

1.1 Introduction.

What is Analysis? It's basically more advanced version of calculus.

Calculus: What's calculus? What's the main object of study? Functions, specifically, real variable & real valued functions, i.e. $f: I \to \mathbb{R}$. What properties of a function do we care about in calculus?

Continuity, differentiability, integrability

$I = [a,b]$, (a,b), $(-\infty, a)$, $(b, +\infty)$ or $(-\infty, +\infty) = \mathbb{R}$

Similarly, in analysis, these are the main topics we care about. What's the difference bewteen calculus like 9A, 9B & Math 150A?

Calculus deals with computational stuff like all kinds of differentiation & integration techniques. But calculus doesn't tell you why can you do it like that.

Analysis tells you why: Analysis care more about why can you do it like that.

For example, ① we all know $\lim_{n\to\infty} \frac{1}{n} = 0$. But why? In fact, what does it mean?

② We all know $f: \mathbb{R} \to \mathbb{R}$ is continuous at x_0 means $\lim_{x\to x_0} f(x) = f(x_0)$? What does it mean? How do know functions like $f(x) = x^n$, a^x, $\log x$, $\sin x$ are continuous?

③. Similarly, what really it means by saying f is differentiable ~~at~~ at $x = x_0$? ~~wha~~ y those functions are differentiable? ~~what~~ why chain rule, product rules hold ture?

④ What does it means by saying $f: [a,b] \to \mathbb{R}$ is integrable on $I = [a,b]$? Why Fundamental Theorem of Calculus holds true? Why some functions are integrable? why some are not?

Strategy of this course:

(I) First, we rigorously define everything, like limits, continuity, differentiability, integrability.

(II) Second, we discuss those definitions rigorous. For example, whqat kind of properties of limits, continuity, differentiation, integration hold & why?

(III) Concrete examples. Not only we will do some computation, but also we do it rigorously. In other words we will justify all our computations like

$$\lim_{n \to \infty} \frac{1}{n} = 0, \quad \lim_{n \to \infty} e^{-n} = 0, \quad \frac{d}{dx} \sin(x^2) = \cos(x^2) \cdot 2x$$

$$\lim_{x \to \pi} \sin(x) = \sin(\pi) = 0$$

Homework, quiz, or exam problems contain both proof based stuff & computational stuff. To be successful in this class, you need to unstand all the notions, how to use their definitions to show their properties, how to apply them rigorously into concrete examples. If you really like math & have thought a lot about calculus, you should have been puzzled by calculus. Then you will like this course. Because we (almost) will not discuss anything without telling you why? This course will give you ideas what is real college math for math major. What is math reasoning? Analysis is also very important for applied math majors.

As we mentioned, the main object of study of both calculus & analysis are functions

$$f : I \to \mathbb{R}$$

where $I \subset \mathbb{R}$ is an interval, e.g. $I = [a,b]$, $(-\infty, a)$, $(b, +\infty)$, (a,b), $(-\infty, \infty) = \mathbb{R}$

Before we study maps from $\mathbb{R}$ to $\mathbb{R}$, we first need to understand what is $\mathbb{R}$, the set of real numbers.

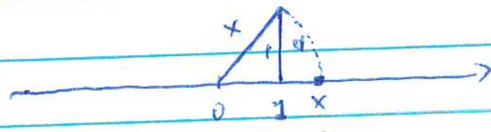
- Algebraically, $\mathbb{R}$ is the set of real numbers such as integers ($\mathbb{Z}$), rational numbers ($\mathbb{Q}$), irrational numbers such as $\sqrt{2}$, e, π.
- Geometrically, $\mathbb{R}$ is a straight line with the origin, length of unit, & direction of positive numbers, i.e. the real line

- Fact: there is a one-to-one correspondence between the set of real numbers & all points on a real line.

~~For instance~~ In other words, for each real number x, there corresponds an ~~pt~~ unique pt on the real line. Conversely, for each pt on the real line, there corresponds a real number. This only holds true for the set of real numbers.

This is actually a very deep & important fact as it builds relation between algebraic objects & geometrical objects.

For example,



There is no number of the form $\frac{m}{n}$ $(m, n \in \mathbb{N})$ that corresponds to this point. Because we know, according to Pythagorean Thm $1^2 + 1^2 = x^2$, i.e. $x^2 = 2$ & $x = \sqrt{2}$ which is a irrational number.

Principles of learning analysis or even any advanced math

(I) In analysis (in fact, in mathematics), one should always try to use geometric stuff to visualize ~~alb~~ algebraic stuff. ~~2~~ as well as should use algebraic ~~stuff~~ objects to accurately (rigorously) describe geometric pictures.

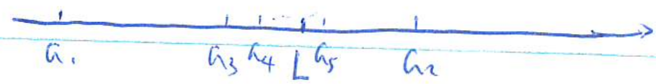
(II) One should always use concrete examples to help understand abstract notions & theorems.

1.2 limits of sequences

Basically, what is $\lim_{n\to\infty} \frac{1}{n}$? More generally, what does it mean by saying a sequence of real numbers $\{a_n, n=1,2,\cdots\}$ converges to a real number L, i.e.

$$\lim_{n\to\infty} a_n = L.$$

- Geometrically,



the sequence a_n move towards L as n goes to infinity.

- A bit more precise, consider intervals (centered) around L with radius ε,

Then no matter how small ε is, eventually, all a_n's are in that interval $(L-\varepsilon, L+\varepsilon)$.

- Rigorously, we define it as following.

Definition 1.1: Let $\{a_n\}, n\in\mathbb{Z}_+$ be a sequence of real numbers. We say $\{a_n\}$ converges to L as n goes to infinity if the following holds:

For any $\varepsilon>0$, there exists a $n_0\in\mathbb{Z}_+$ such that

$$a_n \in (L-\varepsilon, L+\varepsilon) \quad \text{for all } n\geq n_0 \qquad (|a_n - L|<\varepsilon, \text{ for all } n\geq n_0)$$

We express this property in any one of the following notations: $\lim_{n\to\infty} a_n = L$, $\lim_{n} a_n = L$, $a_n \to L$ as $n\to\infty$.

Remarks: (I) This is actually the most important notion in analysis. All other notions such as continuity, differentiability

Suppose $\{a_n\}$ is such that $\exists$ n_0 s.t. $a_n = L$ for all $n \geq n_0$, then $\lim_{n\to\infty} a_n = L$

& integrability are based on the notion of limit. Yet, it's not easy to understand the notion of limit.

(II). Equivalently way to say $\lim_{n\to\infty} a_n = L$ is that no matter how small $\varepsilon > 0$ is, the number of pts a_n outside of the interval $(L-\varepsilon, L+\varepsilon)$ is finite.

(III). If $\{a_n\}$ does not converge (to any number), we say it diverges.

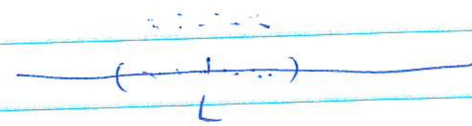
Before more concrete examples, let's discuss two important facts, the proof of which will help you understand the definition.

A sequence $\{b_n\}$ is called a subsequence of $\{a_n\}$ if $b_1 = a_{n_1}$, $b_2 = a_{n_2}$, $\cdots$, $b_k = a_{n_k}$, $\cdots$ where $n_1 < n_2 < n_3 < \cdots < n_k < \cdots$ is an increasing sequence of positive integers. We write the subsequence also in the form $\{a_{n_k}\}$.

Theorem 1.1: If $\lim_{n\to\infty} a_n = L$, then

$$\lim_{k\to\infty} a_{n_k} = L$$

for any subsequence $\{a_{n_k}\}$.



Proof: $\lim_{n\to\infty} a_n = L$ means for any $\varepsilon > 0$, there is a n_0 ~~s.t.~~ such that $|a_n - L| < \varepsilon$ for all $n \geq n_0$.

Now pick an arbitrary subsequence $\{a_{n_k}\}$, there must exist a k_0, s.t. $n_{k_0} \geq n_0$, hence $n_k \geq n_0$ for all $k \geq k_0$. This implies ~~for any $\varepsilon>0$~~

$$|a_{n_k} - L| < \varepsilon, \text{ for all } k \geq k_0$$

To sumarize, for any $\varepsilon > 0$, there exists a k_0 s.t. $|a_{n_k} - L| < \varepsilon$ for all $k \geq k_0$ $\Rightarrow \lim_{k\to\infty} a_{n_k} = L$. $\square$

Theorem 1.2 (Uniqueness of the limit): If $\lim_{n\to\infty} a_n = L$ & $\lim_{n\to\infty} a_n = M$, then $L = M$.

Proof: ~~Sup~~ Argue by contradiction. Suppose it's not true, i.e. $L \neq M$. For ~~this~~ example, we may assume $L < M$

Roughly speaking, we can find ε so small that $L+\varepsilon < M-\varepsilon$ then $(L-\varepsilon, L+\varepsilon) \cap (M-\varepsilon, M+\varepsilon) = \phi$. ~~So~~ But by definition $\lim_{n\to\infty} a_n = L$ & M means $\{a_n\}$ eventually lie in both $(L-\varepsilon, L+\varepsilon)$ & $(M-\varepsilon, M+\varepsilon)$. In particular, there is one $a_n \in (L-\varepsilon, L+\varepsilon) \cap (M-\varepsilon, M+\varepsilon)$, which is not possible.

Rigorously, ~~lim~~ pick $\varepsilon = \frac{M-L}{2}$, then $\lim_{n\to\infty} a_n = L$ implies there exists a n_0, s.t. $|a_n - L| < \varepsilon$ for all $n \geq n_0$ & $\lim_{n\to\infty} a_n = M$ ~~imp~~ means there is a n_1 s.t.

$$|a_n - M| < \varepsilon \quad \text{for all } n \geq n_1$$

Pick any n so that $n \geq n_0$ & $n \geq n_1$, then

$$\begin{cases} |a_n - L| < \varepsilon \\ |a_n - M| < \varepsilon \end{cases} \Rightarrow 2\varepsilon = |L-M| = |L - a_n + a_n - M| \leq |L - a_n| + |a_n - M| < \varepsilon + \varepsilon$$

$\Rightarrow 2\varepsilon < 2\varepsilon$, impossible. The contradiction comes from the assumption that $L \neq M$. So $L = M$. $\square$

Here we used the triangle inequality
$|a+b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$

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Examples: ① Fix $a>0$, then $\lim_{n\to\infty} \frac{1}{n^a} = 0$.

Solution: For any $\varepsilon>0$, we need to find a n_0 so that

$$\left|\frac{1}{n^a} - 0\right| = \frac{1}{n^a} < \varepsilon \quad \text{for all } n \geq n_0$$

Work backwards, when $\frac{1}{n^a} < \varepsilon$

$$\Leftarrow \frac{1}{\varepsilon} < n^a$$

$$\Leftarrow \left(\frac{1}{\varepsilon}\right)^{\frac{1}{a}} < n$$

Pick any n_0 s.t. $n_0 > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{a}}$ (then $\frac{1}{n_0^a} < \varepsilon$),

then for any $n \geq n_0 > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{a}}$

$$\left|\frac{1}{n^a} - 0\right| = \frac{1}{n^a} \leq \frac{1}{n_0^a} < \varepsilon. \quad \square$$

② Fix $0<a<1$, then $\lim_{n\to\infty} a^n = 0$

Solution: For any $\varepsilon>0$, we need to find a n_0 so that

$$|a^n - 0| = a^n < \varepsilon \quad \text{for all } n \geq n_0.$$

$h = \frac{1}{a} - 1 = \frac{1-a}{a}$

$\Uparrow$

let $\frac{1}{a} = 1+h$

$h>0$

$(a+b)^n$

$= a^n + na^{n-1}b$

$+ \binom{n}{2}a^{n-2}b^2$

$+ \cdots$

Work backwards, $a^n < \varepsilon$

$$\Leftarrow \frac{1}{\varepsilon} < \left(\frac{1}{a}\right)^n = (1+h)^n$$

$$\Leftarrow \underline{\frac{1}{\varepsilon} < 1+nh} < (1+h)^n$$

we may assume $\varepsilon<1$

$$\Leftarrow \left(\frac{1}{\varepsilon} - 1\right) < nh$$

$$\Leftarrow n > \frac{1}{h}\left(\frac{1}{\varepsilon} - 1\right)$$

$$\Leftarrow \frac{a}{1-a} n > \frac{a(1-\varepsilon)}{(1-a)\varepsilon}$$

Pick n_0 s.t. $n_0 > \frac{a(1-\varepsilon)}{(1-a)\varepsilon}$ (then $a^{n_0} < \varepsilon \Rightarrow a^n < \varepsilon$, for all $n \geq n_0$)

Then $|a^n - 0| = a^n \leq a^{n_0} < \varepsilon$ $\square$

③ Let $0<\alpha<1$. Then $\lim_n n\alpha^n = 0$.

Proof: Again we let $\frac{1}{\alpha} = 1+h$, $h>0$.

Then $\left(\frac{1}{\alpha}\right)^n = (1+h)^n = 1+nh+\frac{n(n-1)}{2}h^2+\cdots+nh^{n-1}+h^n > \frac{n(n-1)}{2}\cdot h^2$

For each $\varepsilon>0$, we need to find n_0 s.t.

$$n\alpha^n < \varepsilon \quad \text{for all } n \geq n_0$$

work backwards: $n\alpha^n < \varepsilon$

$$\Leftrightarrow \frac{1}{\varepsilon} < \frac{1}{n}\cdot\left(\frac{1}{\alpha}\right)^n = \frac{1}{n}(1+h)^n$$

$$\Leftarrow \frac{1}{\varepsilon} < \frac{1}{n}\frac{n(n-1)}{2}h^2 \quad \left(\text{since } < \frac{1}{n}(1+h)^n\right)$$

$$\Leftrightarrow \frac{1}{\varepsilon} < \frac{(n-1)}{2}h^2$$

$$\Leftrightarrow n-1 > \frac{2}{\varepsilon h^2} \quad \Leftrightarrow n > \frac{2}{\varepsilon h^2}+1$$

Pick a n_0 s.t. $n_0 > \frac{2}{\varepsilon h^2}+1$ $\left(h = \frac{1}{\alpha}-1\right)$,

then for any $n \geq n_0$, $n\alpha^n < \varepsilon$.

④ Is $a_n = (-1)^n$ convergenent?

i.e. $-1, 1, -1, 1, -1, 1, \ldots$

Let $b_k = a_{2k} = (-1)^{2k} = 1$ clearly $\lim_{k\to\infty} b_k = 1$

$c_k = a_{2k-1} = (-1)^{2k-1} = -1$. $\lim_{k\to\infty} c_k = -1$

Both $\{b_k\}$ & $\{c_k\}$ are sub-sequence of $\{a_k\}$.

If $\{a_k\}$ converges to L, then by Thm 1, all its subsequences converges to L. Hence $\{a_n\}$ is not convergent.

$\forall$: for any — s.t. : such that

$\exists$: there exists — WLOG : without loss of generality.

e.g. : for example — i.e.: in other words

1.2 Operations with limits

Thm 1.3. Let $\{a_n\}$ & $\{b_n\}$ be two sequences of real numbers. If $\lim_{n\to\infty} a_n = A$ & $\lim_{n\to\infty} b_n = B$, then $\lim_{n\to\infty}(a_n+b_n)$ exists and equals $A+B$. In other words

$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

Proof: By definition, fix any arbitrary $\varepsilon > 0$

$\lim_{n\to\infty} a_n = A$ ~~means~~ implies ~~$\forall\varepsilon>0$~~, $\exists n_1$, $\forall n \geq n_1$, $|a_n - A| < \frac{\varepsilon}{2}$

$\lim_{n\to\infty} b_n = A$ implies , $\exists n_2$, $\forall n \geq n_2$, $|b_n - B| < \frac{\varepsilon}{2}$

$$\Rightarrow \quad |(a_n+b_n)-(A+B)| = |(a_n-A)+(b_n-B)|$$

$$\leq |a_n - A| + |b_n - B| \; \cancel{< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2}}$$

~~If~~ let $n_0 = \max\{n_1, n_2\}$, if $n \geq n_0$, then

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

To summarize, $\forall \varepsilon > 0$, we found n_0 s.t.

$$|(a_n+b_n)-(A+B)| < \varepsilon, \quad \forall n \geq n_0$$

$$\Rightarrow \lim_{n\to\infty}(a_n+b_n) = A+B \qquad \square$$

We say a sequence $\{a_n\}$ is bounded, if $\exists M > 0$ s.t.

$$|a_n| < M \quad \text{for all } n \geq 1.$$

Thm 1.4 A convergent sequence is bounded.

Proof: By definition, take $\varepsilon = 1$, then $\exists n_0$ s.t.

$$|a_n - A| < 1 \quad \text{for all } n \geq n_0$$

$$\Rightarrow \quad |a_n| - |A| < |a_n - A| < 1 \Rightarrow |a_n| < |A| + 1 \quad \text{for all } n \geq n$$

Let $M = |a_1| + |a_2| + \cdots + |a_{n_0 - 1}| + |A| + 1$.

Then $|a_n| \le M$ for all $n \geq 1$. $\square$

Example: Is $\{a_n\}$ convergent if $a_n = n$?

No, because it's unbounded.

Theorem 1.5. Let $\{a_n\}$ & $\{b_n\}$ be two cequences of real numbers. If $\lim_{n\to\infty} a_n = A$ & $\lim_{n\to\infty} b_n = B$, then $\{(a_n b_n)\}$ is convergent & $\lim_{n\to\infty} a_n b_n = A \cdot B$, i.e.

$$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right)$$

Proof: By Theorem 1.4, $\exists M$ s.t. $|b_n| < M$ for all $n \geq 1$.

Fix an arbitrary $\varepsilon > 0$. By definition,

$\lim_{n\to\infty} a_n = A$ implies $\exists n_1$ s.t.

$$|a_n - A| < \frac{\varepsilon}{2M} \quad \text{for all } n \geq n_1$$

$\lim_{n\to\infty} b_n = B$ implies $\exists n_2$ s.t.

$$|b_n - B| < \frac{\varepsilon}{2(|A|+1)}, \quad \forall\, n \geq n_2$$

Now
$$\begin{aligned}
|a_n b_n - AB| &= |a_n b_n - \cancel{a_n B}\, A b_n + A b_n - AB| \\
&\le |a_n b_n - A b_n| + |A b_n - AB| \\
&= b_n |a_n - A| + A |b_n - B| \\
&\le M |a_n - A| + A |b_n - B| \\
&\le M \cdot \frac{\varepsilon}{2M} + A \cdot \frac{1}{|A|+1} \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Let $n_0 = \max\{n_1, n_2\}$, if $n \geq n_0$, then.

$\square$

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Corollary: $\lim_{n\to\infty}(\lambda a_n) = \lambda \lim_{n\to\infty} a_n$

Proof: set $b_n = \lambda$ for all n. then $\lim_{n\to\infty} b_n = \lambda$.

& $\lim_{n\to\infty}(\lambda a_n) = \lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\cdot\left(\lim_{n\to\infty} b_n\right) = \lambda \lim_{n\to\infty} a_n$ □

Theorem 1.6: Let $\{b_n\}$ be a sequence of real numbers & $\lim_{n\to\infty} b_n = B$ where $B \neq 0$. Then $\exists\, n^*$ s.t.

$$|b_n| \geq \frac{|B|}{2} \quad \text{for all } n \geq n^*$$

Proof: Take $\varepsilon = \frac{|B|}{2}$, then there $\exists\, n^*$ s.t.

$$|b_n - B| < \frac{|B|}{2} \quad \text{for all } n \geq n^*$$

$$\Rightarrow |B| - |b_n| < |b_n - B| < \frac{|B|}{2}$$

$$\Rightarrow |b|_n > |B| - \frac{|B|}{2} = \frac{|B|}{2} \quad \text{for all } n \geq n^*.$$ □

Theorem 1.7: Let $\lim_{n\to\infty} a_n = A$ & $\lim_{n\to\infty} b_n = B$ & $B \neq 0$

Then $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists & equals $\frac{A}{B}$, i.e.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_n a_n}{\lim_n b_n}.$$

Proof: Fix n^* s.t. $|b_n| > \frac{|B|}{2}$ $\forall\, n \geq n_*$.

Fix a $\varepsilon > 0$. Then $\exists\, n_1$ s.t. $|a_n - A| < |B|\frac{\varepsilon}{4}$, $\forall\, n \geq n_1$

$\exists\, n_2$ s.t. $|b_n - B| < |B|^2 \frac{\varepsilon}{4(|A|+1)}$, $\forall\, n \geq n_2$

$$\Rightarrow \left|\frac{a_n}{b_n} - \frac{A}{B}\right| = \left|\frac{a_n B - A b_n}{b_n B}\right| = \frac{|a_n B - AB + AB - A b_n|}{|b_n B|}$$

$$\leq \frac{2}{|B|^2}\left(|B||a_n - A| + |A||B - b_n|\right) \leq \frac{2}{|B|^2}\left(|B|^2\frac{\varepsilon}{4} + |B|^2\frac{|A|}{|A|+1}\frac{\varepsilon}{4}\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

For all n sufficiently large $\iff \exists\, n^*$ large, for all $n \geq n^*$.

Thm 1.8 (Squeeze Thm): Suppose $\{a_n\}$ $\{b_n\}$ & $\{c_n\}$ are ~~sht~~ such that $a_n \leq b_n \leq c_n$ for all n sufficiently large & $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = A$. Then $\lim_{n\to\infty} b_n = A$.

Proof: Let n_* be s.t. $a_n \leq b_n \leq c_n \quad \forall\, n \geq n_*$.
~~Def~~ Fix an arbitrary $\varepsilon > 0$, then by definitions
$\exists\, n_1$, s.t. $|c_n - A| < \varepsilon/3$, $\forall\, n \geq n_1$
$\exists\, n_2$, s.t. $|a_n - A| < \varepsilon/3$, $\forall\, n \geq n_2$
Let $n_0 = \max\{n_*, n_1, n_2\}$. Then $\forall\, n \geq n_0$,

$$
\begin{aligned}
|b_n - A| &= |b_n - c_n + c_n - A| \\
&\leq b_n - c_n + |c_n - A| \\
&\leq (a_n - c_n) + |c_n - A| \\
&= |a_n - c_n| + |c_n - A| \\
&\leq |a_n - A| + |A - c_n| + |c_n - A| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

□

Thm 1.9: Let $\{a_n\}$ & $\{b_n\}$ be two convergent sequences. Suppose $a_n \leq b_n$ for all n ~~tang~~ sufficiently large. Then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$

Proof: Let $A = \lim_n a_n$ & $B = \lim_n b_n$.
Let n_* be that $a_n \leq b_n \quad \forall\, n \geq n_*$.
Then $\forall\, \varepsilon$, $\exists\, n_1$ s.t. $|a_n - A| < \frac{\varepsilon}{2}$
$\exists\, n_2$ s.t. $|b_n - B| < \frac{\varepsilon}{2}$
Let $n_0 = \max\{n_*, n_1, n_2\}$
Then $A < a_n + \frac{\varepsilon}{2} \leq b_n + \frac{\varepsilon}{2} < B + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = B + \varepsilon$

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Thus, we obtain $\forall \varepsilon > 0$, $A \le B + \varepsilon \Rightarrow A \le B$.

Indeed, if $A > B$, then $A > B + \varepsilon$ for $\varepsilon = \frac{A-B}{2}$ $A - B > \frac{A-B}{2} > 0$

contradicts with $A \le B + \varepsilon$ for $\forall \varepsilon > 0$. $\square$.

Application: ① $\lim\limits_{n\to\infty} \frac{n^4+n^3+2n^2}{2n^4+n^2+1} = \lim\limits_{n\to\infty} \frac{1+\frac{1}{n}+\frac{2}{n^2}}{2+\frac{1}{n^2}+\frac{1}{n^4}}$

$\lim(2+\frac{1}{n^2}+\frac{1}{n^4}) = \lim 2 + \lim \frac{1}{n^2} + \lim \frac{1}{n^4} = 2 > 0$

& $\lim$ $= \frac{\lim(1+\frac{1}{n}+\frac{2}{n^2})}{\lim(2+\frac{1}{n^2}+\frac{1}{n^4})} = \frac{1}{2}$

② $a_n = \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+n}}$

Clearly, $\frac{1}{\sqrt{n^2+1}} = \max\{\frac{1}{\sqrt{n^2+1}}, \ldots, \frac{1}{\sqrt{n^2+n}}\}$ &

$\frac{1}{\sqrt{n^2+n}} = \min\{\frac{1}{\sqrt{n^2+1}}, \ldots, \frac{1}{\sqrt{n^2+n}}\}$

$\Rightarrow \frac{n}{\sqrt{n^2+n}} \le a_n \le \frac{n}{\sqrt{n^2+1}}$

But $\lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = \frac{1}{\lim\sqrt{1+\frac{1}{n}}} = 1$

$1 \le \sqrt{1+\frac{1}{n}} \le 1+\frac{1}{n}$ & $\lim(1) = 1$ $\lim(1+\frac{1}{n}) = 1$

$\Rightarrow \lim\sqrt{1+\frac{1}{n}} = 1$

$\lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$

$1 \le \sqrt{1+\frac{1}{n^2}} \le 1+\frac{1}{n^2}$ & $\lim(1+\frac{1}{n^2}) = 1 \Rightarrow \lim\limits_{n\to\infty}\sqrt{1+\frac{1}{n^2}} = 1$

$\Rightarrow \lim\limits_{n\to\infty} a_n = 1$.