## 2.2 Operations with Continuous Functions

Let $f, g : I \to \mathbb{R}$ be two functions defined on a common interval $I$. Then functions
$f \pm g$, $f \cdot g$, and $f/g$ (if $g \neq 0$) $: I \to \mathbb{R}$
are defined by

$$(f \pm g)(x) = f(x) \pm g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x) \qquad \forall x \in \mathbb{R}$$
$$(f/g)(x) = f(x)/g(x) \quad (g(x) \neq 0)$$

Theorem 2.3 (Sum of continuous functions are continuous). Let $f, g : I \to \mathbb{R}$ be continuous at $\xi \in I$. Then $f+g$ is continuous at $\xi$. In particular, if $f$ & $g$ are continuous on $I$, then $f+g$ is continuous on $I$.

Proof: Here we are going to have two different proofs. The first is proof by definition. The second is proof via Theorem 2.2.

First proof: Since $f$ and $g$ are continuous at $\xi$, we have by definition $\forall \varepsilon > 0$.

$\exists \delta_1 > 0$ s.t. $|f(x) - f(\xi)| < \frac{\varepsilon}{2}$ $\forall |x - \xi| < \delta_1$ and $x \in I$;

$\exists \delta_2 > 0$ s.t. $|g(x) - g(\xi)| < \frac{\varepsilon}{2}$ $\forall |x - \xi| < \delta_2$ and $x \in I$.

Set $\delta = \min\{\delta_1, \delta_2\}$. Then if $|x - \xi| < \delta$ and $x \in I$, we have both $|f(x) - f(\xi)| < \varepsilon/2$ and $|g(x) - g(\xi)| < \varepsilon/2$.

Then $\forall\ |x-\xi|<\delta$ and $x\in I$, we have

$$|(f+g)(x)-(f+g)(\xi)| = |f(x)+g(x)-(f(\xi)+g(\xi))|$$
$$= |f(x)-f(\xi)+g(x)-g(\xi)|$$
$$\leq |f(x)-f(\xi)|+|g(x)-g(\xi)|$$
$$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}$$
$$= \varepsilon$$

To sum up. $\forall \varepsilon>0$, we've found a $\delta>0$ s.t.
$|(f+g)(x)-(f+g)(\xi)|<\varepsilon \quad \forall\ |x-\xi|<\delta$ and $x\in I$.
$\Rightarrow \lim\limits_{x\to\xi}(f+g)(x)=(f+g)(\xi)$, i.e. $f+g$ is continuous at $\xi$.

Second proof: By Theorem 2.2, "if $\forall \{x_n\}_{n\geq 1}$ s.t. $x_n\in I\ \forall n\geq 1$ & $\lim\limits_{n\to\infty}x_n=\xi$, we can show

$$\lim_{n\to\infty}(f+g)(x_n)=(f+g)(\xi).$$

then $(f+g)$ is continuous at $\xi$".

To this end, we fix any $\{x_n\}_{n\geq 1}$ s.t. $x_n\in I\ \forall n\geq 1$ & $\lim\limits_{n\to\infty}x_n=\xi$.

By Theorem 2.2 & $f, g$ are continuous at $\xi$, we have $\lim\limits_{n\to\infty} f(x_n) = f(\xi)$

$\lim\limits_{n\to\infty} g(x_n) = g(\xi)$

By sum limit law, we then have

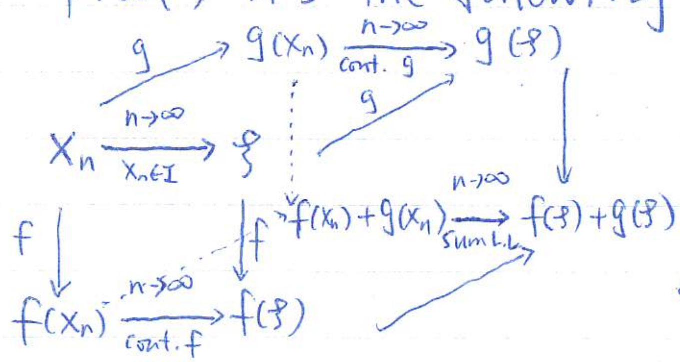
$$\lim_{n\to\infty} (f+g)(x_n) = \lim_{n\to\infty}\big(f(x_n) + g(x_n)\big)$$

$$= \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n)$$

$$= f(\xi) + g(\xi)$$

$$= (f+g)(\xi) \text{ , done!} \qquad \square$$

If we want to use a diagram to illustrate the second proof, it's the following:




$\Rightarrow \lim\limits_{n\to\infty} (f+g)(x_n) = (f+g)(\xi)$

$\Leftrightarrow f+g$ is cont. at $\xi$.

Theorem 2.4 ( Product of continuous functions are continuous)

$f, g : I \to \mathbb{R}$ are cont. at $\xi \in I \Rightarrow f\cdot g$ is cont. at $\xi$.

Proof: Here we are going to give the proof by definition and leave the second proof as a homework problem.

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By continuity of $f$ & $g$ at $\xi$, we have:

$\forall \varepsilon > 0$, $\exists \delta_1 > 0$ s.t. $|f(x) - f(\xi)| < \boxed{\frac{1}{2|f(\xi)|+2}} \varepsilon \quad \forall |x-\xi| < \delta_1, x \in I$.

$\exists \delta_2 > 0$ s.t. $|g(x) - g(\xi)| < \boxed{\frac{1}{2|g(\xi)|+1}} \cdot \varepsilon \quad \forall |x-\xi| < \delta_2, x \in I$

Here similar to the proof of product limit law, we need a little more fact:

Again by continuity of $f$ at $\xi$, we can find $\delta_0 > 0$ s.t. $|f(x) - f(\xi)| < 1 \quad \forall |x - \xi| < \delta_0, x \in I$.

By triangle inequality, $|f(x)| - |f(\xi)| \le |f(x) - f(\xi)| < 1$

$$\Rightarrow |f(x)| \le |f(\xi)| + 1, \ \forall |x-\xi| < \delta_0, x \in I.$$

Now we set $\delta = \min\{\delta_0, \delta_1, \delta_2\}$, then $\forall |x-\xi| < \delta$ & $x \in I$, we have

$$\begin{aligned}
|(f\cdot g)(x) - (f\cdot g)(\xi)| &= |f(x)\cdot g(x) - f(\xi)\cdot g(\xi)| \\
&= |f(x)\cdot g(x) - f(x)\cdot g(\xi) + f(x)\cdot g(\xi) - f(\xi)\cdot g(\xi)| \\
&\le |f(x)\cdot g(x) - f(x)\cdot g(\xi)| + |f(x)\cdot g(\xi) - f(\xi)\cdot g(\xi)| \\
&= |f(x)|\cdot|g(x) - g(\xi)| + |g(\xi)|\cdot|f(x) - f(\xi)| \\
&\le (|f(\xi)|+1)\frac{\varepsilon}{2(|f(\xi)|+1)} + |g(\xi)|\frac{\varepsilon}{2|g(\xi)|+1} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$, s.t. $|(f\cdot g)(x) - (f\cdot g)(\xi)| < \varepsilon \quad \forall |x-\xi| < \delta, x \in I$ $\square$

Theorem 2.5 (Quotient of continuous functions are continuous)

$f, g: I \to \mathbb{R}$ cont. at $\xi$ and $g(\xi) \neq 0 \Rightarrow \frac{f}{g}$ is cont. at $\xi$.

Proof: Here we use the second proof (by Thm 2.2) and leave the proof by definition as a homework problem.

By Thm 2.2, $\frac{f}{g}$ is cont. at $\xi$ if:

"$\forall \{x_n\}$ s.t. $x_n \in I$ $\forall n \geq 1$ & $\lim_{n\to\infty} x_n = \xi$, we have $\lim_{n\to\infty} \left(\frac{f}{g}\right)(x_n) = \left(\frac{f}{g}\right)(\xi)$." Fix such a $\{x_n\}_{n\geq 1}$.

Since $f$ & $g$ are cont. at $\xi$ & $g(\xi) \neq 0$, by Thm 2.2 we have

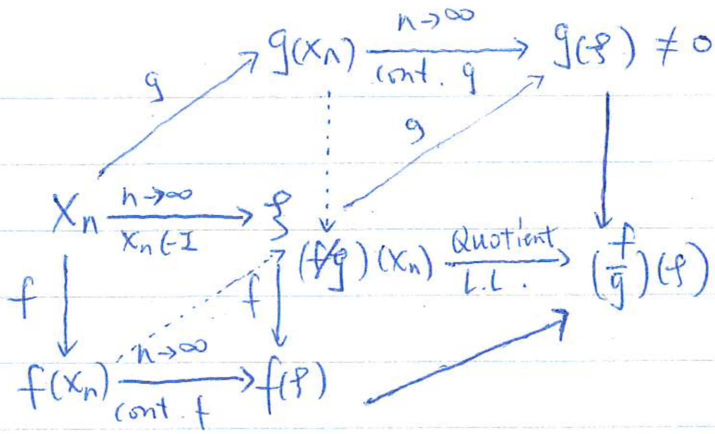
$$\lim_{n\to\infty} f(x_n) = f(\xi)$$
$$\lim_{n\to\infty} g(x_n) = g(\xi)$$

By quotient limit law, we have

$$\lim_{n\to\infty} \left(\frac{f}{g}\right)(x_n) = \lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{\lim f(x_n)}{\lim g(x_n)} = \frac{f(\xi)}{g(\xi)} = \left(\frac{f}{g}\right)(\xi).$$

done! $\square$

use a diagram:




$\forall \{x_n\}, \lim x_n = \xi$
$x_n \in I, \forall n \geq 1,$

$\Rightarrow \lim\limits_{n\to\infty} \left(\frac{f}{g}\right)(x_n)$

$= \lim\limits_{n\to\infty} \left(\frac{f}{g}\right)(\xi)$

Definition 2.3 (composition of functions)

Let $g: I \to \mathbb{R}$ be a function. Let $J_0 = \{g(x) : x \in I\}$, i.e. the range of $g$. Let $J$ be an interval s.t. $J_0 \subseteq J$. Let $f: J \to \mathbb{R}$ be another function. Then for each $x \in I$, there corresponds a number

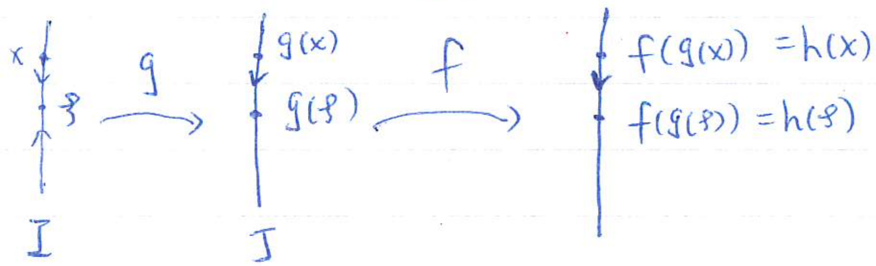
$$f(g(x)) \in \mathbb{R}.$$

Thus we can define a new function $h(x) := f(g(x))$ on $I$. It's called the composition of $f$ with $g$, denoted by $h = f \circ g$.

Theorem 2.6 (composition of continuous functions are continuous)

Let $f$ and $g$ be as in def 2.3. Suppose $g$ is continuous at $\xi \in I$ & $f$ is continuous at $g(\xi) \in J$. Then $h(x) = f \circ g(x)$ is continuous at $\xi \in I$.

Proof: Again we may have two different proofs. Geometrically,




First proof by definition:

$f$ is cont. at $g(\xi) \Rightarrow \forall \varepsilon>0, \exists \eta>0$ s.t.

$$|f(y)-f(g(\xi))|<\varepsilon \quad \forall\, |y-g(\xi)|<\eta \;\&\; y\in J$$

$g$ is cont. at $\xi \Rightarrow$ For the $\eta>0$ above, $\exists \delta>0$ s.t.

$$|g(x)-g(\xi)|<\eta, \quad \forall\, |x-\xi|<\delta \;\&\; x\in I.$$

Clearly $g(x)\in J$ $\forall x\in I$ by definition.

Thus $\forall \varepsilon>0$, $|f(g(x))-f(g(\xi))|<\varepsilon \quad \forall\, |x-\xi|<\delta \;\&\; x\in I$

$\Rightarrow \underset{h(x)=}{f(g(x))}$ is continuous at $\xi$.

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Second proof: Fix any $\{x_n\}_{n\geq 1}$ s.t. $x_n\in I$ $\forall n\geq 1$ & $\lim\limits_{n\to\infty} x_n=\xi$

By continuity of $g$ & Thm 2.2 $\Rightarrow \lim\limits_{n\to\infty} f(x_n)=g(\xi)$

$g(x_n)\in J,\ \forall n\geq 1$

By continuity of $f$ & Thm 2.2

$\Rightarrow \lim\limits_{n\to\infty} f(g(x_n)) = \lim\limits_{n\to\infty} f(g(\xi)) \Rightarrow f\circ g$ is cont. at $\xi$ by Thm 2.2 $\square$

$$\left(\begin{array}{ccc} x_n & \xrightarrow{n\to\infty} & \xi \\ g\downarrow & & \downarrow \\ g(x_n) & \xrightarrow[\text{cont. } g]{n\to\infty} & g(\xi) \\ f\downarrow & & \downarrow f \\ f(g(x_n)) & \xrightarrow[\text{cont. } f]{n\to\infty} & f(g(\xi)) \end{array}\right) \Rightarrow \begin{array}{ccc} x_n & \xrightarrow{n\to\infty} & \xi \\ f\circ g \downarrow & & \downarrow f\circ g \\ f\circ g(x_n) & \xrightarrow{n\to\infty} & f\circ g(\xi) \end{array} \Rightarrow f\circ g \text{ is cont. at } \xi.$$

Application of all the theorems:

First we note that we can extend the sum & product of two functions to the sum & product of any finite number of functions.

Example 1: Recall we proved by definition that $f(x) = c, x, x^2$ are continuous.

① $f(x) = x^n$, $n \geq 0$ is continuous on $\mathbb{R}$.

Proof: $f(x) = \underbrace{x \cdot x \cdots x}_{n}$ is a product of $n$ functions $g(x) = x$ which is continuous $\Rightarrow f(x)$ is continuous.

② Consider $p(x) = \sum_{k=0}^{n} a_k \cdot x^k$ a polynomial of degree $n$. Here $a_0, \dots, a_n$ are ~~contl~~ constants. $p(x)$ is continuous on $\mathbb{R}$.

Proof: For each $k$, $a_k \cdot x^k$ is continuous since it's a product of $f(x) \equiv a_k$ & $g(x) = x^k$ which are continuous. Thus

$$p(x) = a_0 + a_1 \cdot x + \cdots + a_n x^n$$

is continuous on $\mathbb{R}$ since it's a finite sum of continuous functions.

③ Even more general, we call $r(x) = \frac{p(x)}{q(x)}$ a rational function if both $p$ & $q$ are polynomials. Note $r$ is not defined on such $x$'s where $q(x) = 0$. But $r(x)$ is continuous at every point where it's well-defined.

Proof: At every point $x$ where $r$ is well defined, it's a quotient of two continuous functions with nonzero denominator. Thus it's continuous at every such $x$.

Example 2: $\tan(x) = \frac{\sin(x)}{\cos(x)}$ is continuous at every $x \neq n\pi + \frac{\pi}{2}$. $f(x) = \sin(x^2)$ is continuous on $\mathbb{R}$. Similarly $f(x) = \sin(\frac{1}{x})$ is continuous at every $x \neq 0$. $f(x) = \sin(\sqrt{x}) : [0, +\infty) \to \mathbb{R}$ is cont.

Proof: we just need to apply all the Theorems 2.3 - 2.6. $\square$

Example 3: Compute $\lim_{n \to \infty} \sin(\frac{1}{\sqrt{n+1}})$.

Solution. Note $\sin(\frac{1}{\sqrt{n+1}}) = \sin\sqrt{\frac{1}{n+1}}$.

We know from example 2, $f(x) = \sin\sqrt{x} : [0, +\infty) \to \mathbb{R}$ is continuous on $[0, +\infty)$.

Thus by Theorem 2.2, $\forall \{x_n\}_{n \geq 1}$ $x_n \geq 0$ $\lim_{n \to \infty} x_n = 0$ we have $\lim_{n \to \infty} f(x_n) = f(0)$.

Now we know $\lim_{n \to \infty} \frac{1}{n+1} = 0$ & $\frac{1}{n+1} \geq 0$ $\forall n \geq 1$.

$\Rightarrow \lim_{n \to \infty} \sin(\frac{1}{\sqrt{n+1}}) = \lim_{n \to \infty} f(\frac{1}{n+1}) = f(0) = \sin\sqrt{0} = 0$

$\square$