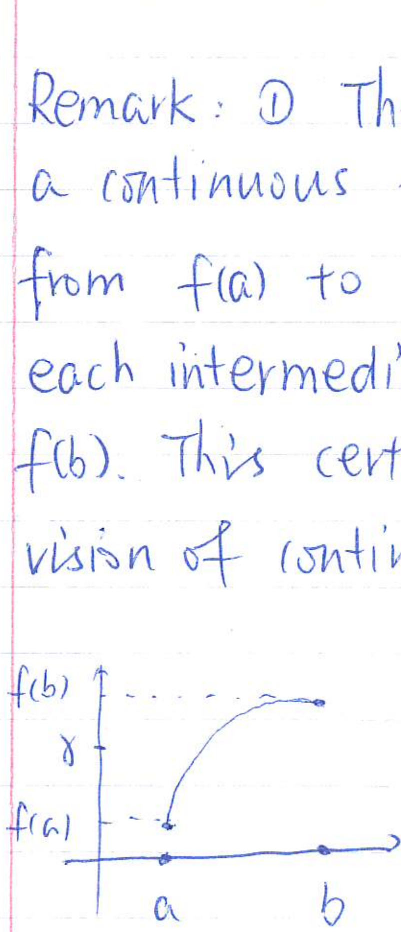
2.4. Intermediate Values

The main goal of this section is to prove the following Intermediate Value Theorem.

Theorem 2.9. Let $f: I \to \mathbb{R}$ be a continuous function defined on a closed, bounded interval $I = [a,b]$. Assume $f(a) \neq f(b)$ and Let γ be an any number between $f(a)$ & $f(b)$. Then there exists at least one point $c \in (a,b)$ s.t. $f(c) = \gamma$.

Remark: ① Theorem 2.9 basically says that a continuous function f may not go from $f(a)$ to $f(b)$ without passing through each intermediate values between $f(a)$ & $f(b)$. This certainly matches the geometric vision of continuity



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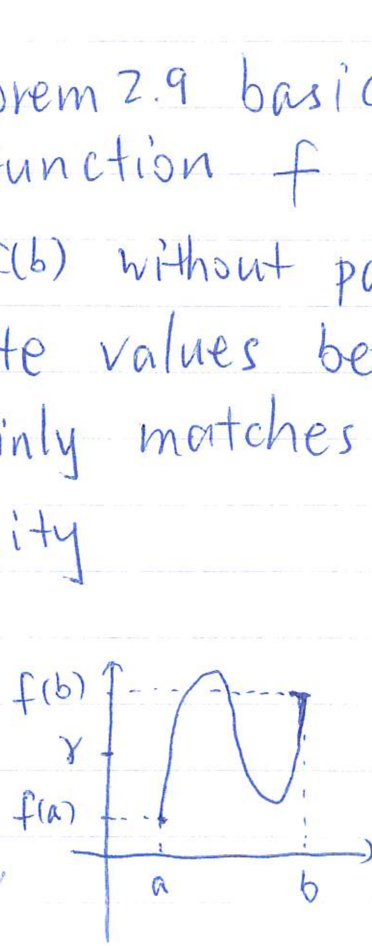

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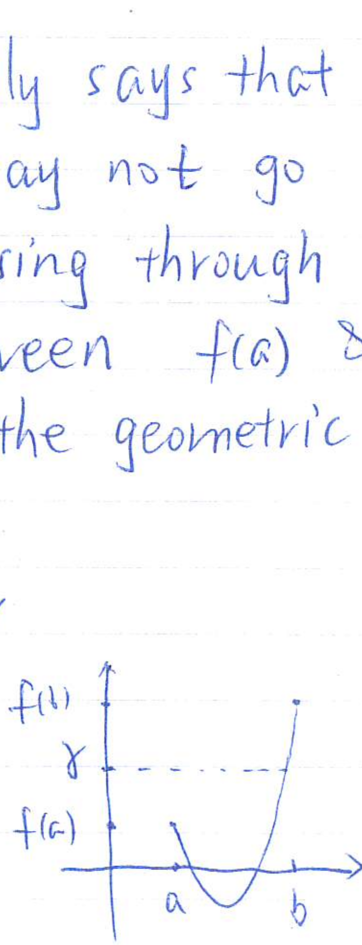


② Note that if we take any x_1 & x_2 s.t.

$$a < x_1 < x_2 < b,$$

then f is continuous function on the smaller interval $[x_1, x_2]$. Thus we can apply Thm 2.9 to $f : [x_1, x_2] \to \mathbb{R}$ and obtain:

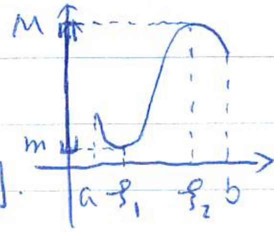
if $f(x_1) \neq f(x_2)$, then $f(x)$ must assume all values between $f(x_1)$ and $f(x_2)$

③ In particular, for f as in Thm 2.9, by Theorem 2.8, f has at least one minimum point ξ_1 where f assumes its minimum m & has at least one maximum point ξ_2 where f assumes its maximum M, i.e.

$f(\xi_1) = m$ is the minimum value of f on I

$f(\xi_2) = M$ is the maximum value of f on I

Then by Thm 2.9, f must assume all values between m & M. Thus $f(I) = [m, M]$, i.e. a closed, bounded interval.



We may formulate it as a corollary.

Corollary: Let $f: I \to \mathbb{R}$ be continuous on $I = [a, b]$. Then $f(I) = [m, M]$ where m is the minimum of f & M is the maximum of f.

Proof of Thm 2.9: Without loss of generality, we assume $f(a) < f(b)$. Set $I_1 = I = [a, b]$. Divide I_1 into two sub-intervals with equal length by introducing its mid-point $c_1 = \frac{a+b}{2}$. Take any $f(a) < \gamma < f(b)$

If $f(c_1) = \gamma$, then we are done; otherwise:

case (I): if $f(c_1) < \gamma$, then we set $I_2 = [c_1, b]$,

$$\text{Thus } f(c_1) < \gamma < f(b)$$

case (II): if $f(c_1) > \gamma$, then we set $I_2 = [a, c_2]$.

$$\text{Thus } f(a) < \gamma < f(c_1)$$

In any case, we have $f(\text{left end-pt of } I_2) < \gamma < f(\text{right end-pt of } I_2)$.

Then we repeat the same process for I_2 by introducing its mid-pt. c_2. Then either

$f(c_2) = \gamma$, we are done;

or we obtain I_3 which is half of I_2 s.t.

$$f(\text{left end-pt of } I_3) < \gamma < f(\text{right end pt of } I_3)$$

By induction, either we find at step n a c_n with $a < c_n < b$ s.t. $f(c_n) = \gamma$; or we construct a nest

$$I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n \supset \cdots$$

s.t.: if we set $I_n = [a_n, b_n]$, $\forall n \geq 1$, then

(i) $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \quad \forall n \geq 1$

(ii) $b_n - a_n = \frac{b-a}{2^{n-1}}$

(iii) $f(a_n) < \gamma < f(b_n)$, $\forall n \geq 1$.

By Theorem 1.15 and its proof regarding nest, we have $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$ and $\lim_{n\to\infty} a_n = \xi = \lim_{n\to\infty} b_n$

By continuity of f and Thm 2.2, we have

$$\lim_{n\to\infty} f(a_n) = f(\xi) = \lim_{n\to\infty} f(b_n)$$

By squeeze thm & (iii) above, we must have $\gamma = f(\xi)$. Clearly $\xi \in I$ & $\xi \neq a, b$ since $f(a) < \gamma = f(\xi) < f(b)$. Thus $\xi \in (a, b)$ & $f(\xi) = \gamma$ as desired. $\square$