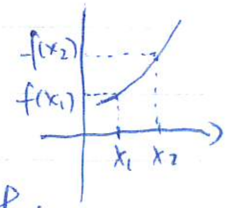
2.5 Monotone Functions and Inverse Functions

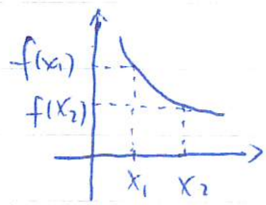
Definition 2.6. Let $f: I \to \mathbb{R}$ be a function. We say f is

(I) monotone increasing on I if:
$f(x_1) \le f(x_2) \quad \forall\, x_1 < x_2 \in I$



(II) strictly monotone increasing on I if:
$f(x_1) < f(x_2) \quad \forall\, x_1 < x_2 \in I$

(III) monotone decreasing on I if:
$f(x_1) \ge f(x_2) \quad \forall\, x_1 < x_2 \in I$



(IV) Strictly monotone decreasing on I if:
$f(x_1) > f(x_2) \quad \forall\, x_1 < x_2 \in I$

A (strictly) monotone function is a function that is either (strictly) monotone increasing or (strictly) monotone decreasing.

Definition 2.7. Consider a function $f: I \to \mathbb{R}$. We say f is 1-to-1 if $f(x_1) \ne f(x_2) \ \forall\, x_1 \ne x_2 \in I$

Let $J = f(I)$. Then $\forall\, y \in J$, $\exists$ a unique $x \in I$ s.t.
$$f(x) = y$$
Thus we may define a new function $g: J \to I$

such that $g(y) = x$, where $f(x) = y$. (*)
Such function g is called the inverse of f, denoted $g = f^{-1}$. By (*), we have

$g(f(x)) = g(y) = x \Rightarrow g \circ f : I \to I$ is the identity map

$f(g(y)) = f(x) = y \Rightarrow f \circ g : J \to J$ is the identity map as well.

Lemma: If $f: I \to \mathbb{R}$ is strictly monotone, then f is 1-to-1. Moreover, if f is strictly monotone increasing (~~ded~~decreasing, resp.), then $g \overset{=f^{-1}}{}$ is strictly monotone increasing (decreasing, resp.)

Proof: We consider the case where f is strictly monotone increasing. The case where f is decreasing is similar.

By definition, $f(x_1) < f(x_2)$ $\forall x_1 < x_2 \in I$.

Take any $x_1 \neq x_2$, then either $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

or $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$

In both cases, $f(x_1) \neq f(x_2) \Rightarrow f$ is 1-to-1.

$\Rightarrow g = f^{-1}$ exists

Take any $y_1 < y_2 \in J = f(I)$. Then there are x_1 & $x_2 \in I$ s.t. $f(x_1) = y_1$ & $f(x_2) = y_2$

Then either $x_1 > x_2$ or $x_1 < x_2$. But $x_1 > x_2$ implies $y_1 > y_2$. Thus $x_1 < x_2$. By definition

$$x_1 = g(y_1) < g(y_2) = x_2 \qquad \forall\ y_1 < y_2 \in J$$

$\Rightarrow$ g is strictly monotone increasing.

Theorem 2.10: Let f be a strictly monotone, continuous function on a closed, bounded interval $I = [a, b]$. Then its inverse $g = f^{-1}$ is strictly monotone and continuous on $J = f(I)$.

Proof: Again we assume that f is strictly montone increasing. The other case can be done similarly.

Then by Corollary of Section 2.4, we know that $J = f(I) = [f(a), f(b)]$. By Lemma, we know that $g = f^{-1} : [f(a), f(b)]$ is strictly monotone increasing. Thus to prove Thm 2.10, we only need to show that g is continuous on $[f(a), f(b)]$

So we need to show that g is continuous at every $\gamma \in [f(a), f(b)]$. We shall focus on the case $\gamma \in (f(a), f(b))$. The cases with end-points can be done similar and will be left as a homework ~~to~~ problem.

From now on, we fix a $\gamma \in (f(a), f(b))$ & will show g is continuous at γ. First we note that $g(\gamma) = \xi \in (a, b)$. Note $f(\xi) = \gamma$.

By definition of continuity, we need to show: $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $|g(y) - g(\gamma)| < \varepsilon$

$$\forall\, |y - \gamma| < \delta \ \&\ y \in J.$$

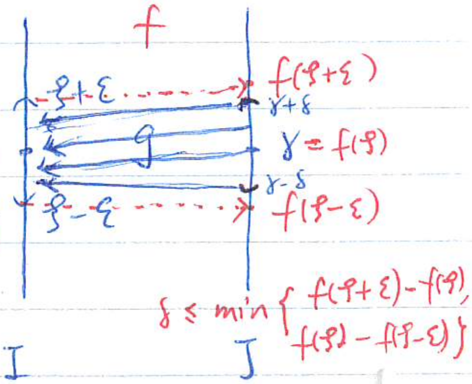
Note we only need to consider very small ε. Indeed in the definition, if δ works for ε, then it works for all $\varepsilon' > \varepsilon$.

Thus we fix any $\varepsilon > 0$ s.t. $[\xi - \varepsilon, \xi + \varepsilon] \subset I$.

Since $\xi - \varepsilon < \xi < \xi + \varepsilon \in I$

$\Rightarrow f(\xi - \varepsilon) < f(\xi) < f(\xi + \varepsilon)$

Set $\delta = \min\{f(\xi+\varepsilon) - f(\xi),\ f(\xi) - f(\xi - \varepsilon)\}$



Then

$$f(\xi - \overset{\varepsilon}{\cancel{\gamma}}) \le \gamma - \delta < \gamma < \gamma + \delta \le f(\xi + \varepsilon) \qquad (*)$$

In particular $(\gamma - \delta, \gamma + \delta) \subset J$. Moreover,

$y \in (\gamma - \delta, \gamma + \delta) \iff |y - \gamma| < \delta \iff \gamma - \delta < y < \gamma + \delta$

By monotonicity of g, we have

$$g(f(\xi-\varepsilon)) \le g(\gamma-\delta) < g(y) < g(\gamma+\delta) \le g(f(\xi+\varepsilon))$$

$$\Rightarrow \xi-\varepsilon < g(y) < \xi+\varepsilon$$

$$\Leftrightarrow |g(y)-\xi| < \varepsilon \Leftrightarrow |g(y)-g(\gamma)| < \varepsilon.$$

To sum up, $\forall \varepsilon > 0$ small, we've found a $\delta > 0$ s.t. $|g(y)-g(\gamma)| < \varepsilon \quad \forall\ |y-\gamma| < \delta$

$\Rightarrow g$ is continuous at γ. $\square$

Question: Where did we use the continuity of f?

Examples

① Consider $f(x)=x^2 : [0,+\infty) \to [0,+\infty)$. It's inverse $g : [0,+\infty) \to [0,+\infty)$ is continuous & strictly monotone increasing.

Proof: Clearly, $f(x)=x^2$ is continuous and strictly monotone increasing on $[0,+\infty)$. Moreover $\forall y \in [0,+\infty)$, there is a ~~unique~~ $n \in \mathbb{Z}^+$ s.t. $n = m^2 > y$

Thus $y \in [0, \overset{n}{\cancel{m}}] = f([0,m])$ by corollary of Thm 2.9.

$\Rightarrow \exists x \in [0,m]$ s.t. $f(x)=y$. Since y is arbitrarily chosen, we have $f([0,+\infty)) = [0,+\infty)$. Thus the inverse $g = f^{-1}$ is defined on $[0,+\infty)$.

To show g is strictly monotone increasing, take any $y_1 < y_2 \in [0,+\infty)$. Then we may consider $g: [0, M] \to [0, g(M)]$ for some $M > y_2$. Then $g|_{[0,M]}$ is the inverse of $f|_{[0,g(M)]}$. Thus Thm 2.10 implies that g is strictly monotone increasing on $[0,M] \Rightarrow g(y_1) < g(y_2)$.

Since $y_1 < y_2$ are abitrarily chosen
$\Rightarrow g$ is strictly monotone increasing on $[0,+\infty)$.

To show g is continuous on $[0,+\infty)$. Take any $\gamma \in [0,+\infty)$. We may again restrict g to $[0, M]$ for some $M > \gamma$. Then $g|_{[0,M]}$ is the inverse of the continuous function $f|_{[0,g(M)]} \Rightarrow g$ is continuous on $[0, M]$
$\Rightarrow g$ is continuous at $\forall \gamma \Rightarrow g$ is continuous on $[0,+\infty)$ since γ is abitrarily chosen.

We denote $g(x) = \sqrt{x}$ $\forall x \in [0,+\infty)$.

② Through the same process, we can show for any $n \in \mathbb{Z}_+$, there is a continuous & strictly monotone increasing function

$g: [0,+\infty) \to [0,+\infty)$ which inverts

$f: [0,+\infty) \to [0,+\infty)$ where $f(x) = x^n$.

We denote $g(x) = x^{\frac{1}{n}}$.

③. By ② we may define $\forall\, n, m \in \mathbb{Q}_+$, $\forall x \in \mathbb{R}_+ \cup \{0\}$

$$x^{\frac{m}{n}} = \underbrace{x^{\frac{1}{n}} \cdot x^{\frac{1}{n}} \cdots x^{\frac{1}{n}}}_{m \text{ times}}.$$ Thus

$g(x) = x^{\frac{m}{n}}: [0,+\infty) \to [0,+\infty)$ is continuous & strictly monotone increasing, i.e. $g(x) = x^r$ is a well-defined cont. & strictly monotone increasing function from $[0,+\infty)$ to $[0,+\infty)$.

④ If $r \in \mathbb{Q}_-$, i.e. a negative rational number, we define $g(x) = x^r := \frac{1}{x^{-r}}$ which is well-define $\forall x > 0$ since $-r \in \mathbb{Q}_+$. Since x^{-r} is continuous & increasing on $(0,+\infty)$

$\Rightarrow g(x) = \frac{1}{x^{-r}} : (0,+\infty) \to (0,+\infty)$ is cont. & decreasing.

Now we've defined $g(x) = x^r$ for all $r \in \mathbb{Q}$ and have shown their continuity on their domain.

⑤ We know that $\overset{f(x)}{\sin(x)} : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is strictly monotone ~~&~~ increasing & continuous. Thus it has an inverse

$$\sin^{-1}(x) : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$$

sometimes $\sin^{-1}(x) = \arcsin(x)$

which is again cont. & strictly monotone increasing.

⑥ We know that $f(x) = \tan(x) : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is cont. & strictly monotone increasing.

Thus $\tan^{-1}(x) : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is cont. & strictly monotone increasing.

sometimes $\tan^{-1}(x) = \arctan(x)$