1.3 Monotone Sequences

Definition 1.2: ① A sequence $\{a_n\}$ of real numbers is called monotone increasing if $a_n \le a_{n+1}$ for all n. It is strictly monotone increasing if $a_n < a_{n+1}$ for all n.

② A sequence $\{a_n\}$ is called monotone decreasing if $a_n \ge a_{n+1}$ for all n. It's strictly monotone decreasing if $a_n > a_{n+1}$ for all n.

③ A (strictly) monotone sequence is a sequence that is either (strictly) monotone increasing or (strictly) monotone decreasing.

Theorem 1.10. If a monotone increasing sequence is bounded above, then it is convergent.

Axiom of real number: $\mathbb{R}$ satisfies the least-upper-bound propety, i.e. let $S \subset \mathbb{R}$ be a subset bounded from above (it means there is a $M \in \mathbb{R}$ s.t. $x \le M$ for all $x \in S$). Then there exists a $\alpha \in \mathbb{R}$ s.t. the following holds true:

(i) α is an upper bound of S, i.e. $x \le \alpha$ for all $x \in S$.

(ii) for any $\varepsilon > 0$, $\alpha - \varepsilon$ is not an upper bound of S, i.e. for any $\varepsilon > 0$, there is a $x_0 \in S$ s.t. $x_0 > \alpha - \varepsilon$.

Such a α is called a the least upper bound, or supremum, of S. denoted by $\alpha = \sup S$.

Proof of Thm 1.10: let $\{a_n\}$ be the monotone increasing sequence that is bounded above. Let $A = \sup\{a_n\}$.

Then we claim $A = \lim_{n\to\infty} a_n$. Indeed, since $A = \sup\{a_n\}$, for any $\varepsilon > 0$, there is a n_0 s.t. $a_{n_0} > A - \varepsilon$. $\{a_n\}$ is monotone increasing implies for all $n \geq n_0$

$$a_n \geq a_{n_0} > A - \varepsilon \qquad ①$$

On the other hand, A is an upper bound of $\{A_n\}$ implies

$$a_n \leq A \quad \text{for all } n \geq 1. \qquad ②$$

① & ② together implies for all $n \geq n_0$

$$A - \varepsilon < a_n \leq A$$

$\Rightarrow \quad |a_n - A| < \varepsilon \quad$ for all $n \geq n_0$

In summary, for any $\varepsilon > 0$, we found a $n_0 \in \mathbb{Z}_+$ s.t.

$$|a_n - A| < \varepsilon \quad \text{for all } n \geq n_0$$

$\Rightarrow \quad \lim_{n\to\infty} a_n = A.$ □

Corollary: If $\{a_n\}$ is monotone decreasing ~~then $\{a_n\}$~~ and is bounded below, then it is convergent.

Proof: $\{a_n\}$ is monotone decreasing and is bounded below. Then $\{-a_n\}$ is monotone increasing and is bounded above. Indeed,

- $a_n \geq a_{n+1} \Rightarrow -a_n \leq -a_{n+1}$ for all n

  $\Rightarrow \{-a_n\}$ is monotone increasing

- $a_n \geq M$ for all n $\Rightarrow -a_n \leq -M$ for all n

  $\Rightarrow \{-a_n\}$ is bounded above

Now by Thm 1.10, $\{-a_n\}$ is convergent. By ~~Thm~~ Corollary of Thm 1.5, it implies $\{a_n\}$ is convergent. □

Theorem 1.11. The sequence
$$b_n = \left(1+\frac{1}{n}\right)^n$$
is monotone increasing and $2 \le b_n < 3$.

Proof: By binomial expansion,
$$\left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{1}{n}\right)^k$$
$$+ \cdots + \frac{n!}{n!}\left(\frac{1}{n}\right)^n$$

Clearly, $\dfrac{n(n-1)\cdots(n-k+1)}{n^k} = \dfrac{n-1}{n}\cdot\dfrac{n-2}{n}\cdots\dfrac{n-k+1}{n} = \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$

$$\Rightarrow \left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots +$$
$$\frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) \qquad (*)$$

Similarly
$$\left(1+\frac{1}{n+1}\right)^{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots +$$
$$\frac{1}{k!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right) + \cdots + \cancel{\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)}\ \frac{1}{n!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)$$
$$+ \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right) \qquad (**)$$

Clearly, $\dfrac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) < \dfrac{1}{k!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right)$

for all $2 \le k \le n$

$\Rightarrow \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} \Rightarrow b_n < b_{n+1}$ for all n, i.e. $\{b_n\}$ is strictly monotone increasing.

By $(*)$, $b_n = \left(1+\frac{1}{n}\right)^n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$

Clearly, $n! = n(n-1)\cdots 2\cdot 1 \geq \underbrace{2\cdots 2}_{n-1} = 2^{n-1}$ for all n.

(18)

$\Rightarrow$

$b_n < 1 + 1 + \frac{1}{2} + (\frac{1}{2})^2 + \cdots + (\frac{1}{2})^{n-1}$

$= 1 + \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = 1 + 2 - (\frac{1}{2})^{n-1} < 1+2 = 3$

Clearly $b_n \geq 2$ for all n. □

By Thm 1.10, $\lim_{n\to\infty} b_n$ exists & $\lim_{n\to\infty} b_n < 3$.

The limit is denoted by e, i.e. $e = \lim_{n\to\infty}(1+\frac{1}{n})^n$.

Defition: ① We say $\{a_n\}$ ~~diverges~~ converges to $+\infty$ if for any $N \in \mathbb{Z}_+$, there exists a n_0 s.t.

$$a_n \geq N \quad \text{for all } n \geq n_0$$

It's denoted by $\lim_{n\to\infty} a_n = +\infty$, or $a_n \to +\infty$ as $n \to +\infty$.

② $\{a_n\}$ ~~diverges to~~ converges to $-\infty$, if for any $N \in \mathbb{Z}_+$ there exists a n_0 s.t.

$$a_n \leq -N, \quad \text{for all } n \geq n_0$$

It's denoted as $\lim_{n\to\infty} a_n = -\infty$, or $a_n \to -\infty$ as $n \to -\infty$.

Theorem 1.12. A monotone increasing sequence is either convergent to a real number or converges to $+\infty$.

Proof: If $\{a_n\}$ is bounded above, then Thm 1.10 implies $\lim_{n\to\infty} a_n = A$, where $A = \sup\{a_n\}$.

If $\{a_n\}$ is not bounded above, then for any $N \in \mathbb{Z}_+$, there is a n_0 s.t. $a_{n_0} \geq N$. Thus

$a_n \geq N$ for all $n \geq n_0$ (as $a_n \geq a_{n_0}$ for all $n \geq n_0$)

$\Rightarrow \lim_{n\to\infty} a_n = +\infty$

1.4 Bolzano-Weierstrass Theorem

Definition 1.3: Let $S \subset \mathbb{R}$ be a subset. We say $\alpha \in \mathbb{R}$ is a limit point of S if the following holds true:

for any $\varepsilon > 0$, there is a number $a \neq \alpha$, $a \in S$ s.t.

$$|a - \alpha| < \varepsilon.$$

We may also say α is a cluster point or a point of accumulation of S.

Theorem 1.13. If α is a limit point of S, then there exists a sequence $\{a_n\}$ of mutually distinct points that belong to S such that $\lim\limits_{n\to\infty} a_n = \alpha$.

Proof: By definition 1.3, for $\varepsilon_1 = 1$, we can find a $a_1 \neq \alpha$, $a_1 \in S$

s.t. $|a_1 - \alpha| < 1$

Then for $\varepsilon_2 = \min\{\frac{1}{2}, |a_1 - \alpha|\}$, we can find a $a_2 \neq \alpha$, $a_2 \in S$ s.t. $|a_2 - \alpha| < \varepsilon_2$

Since $|a_2 - \alpha| < \varepsilon_2 \leq |a_1 - \alpha| \Rightarrow a_1 \neq a_2$

Proceed like this, then for each n, and for

$$\varepsilon_n = \min\{\frac{1}{n}, |a_1 - \alpha|, \ldots, |a_{n-1} - \alpha|\}$$

we can find a $a_n \neq \alpha$, $a_n \in S$ s.t.

$$|a_n - \alpha| < \varepsilon_n$$

Then since $|a_n - \alpha| < \varepsilon_n \leq |a_i - \alpha|$ for all $i = 1, \ldots, n-1$

$\Rightarrow a_n$ is different from $a_1, \ldots, a_{n-1}$

Moreover $|a_n - \alpha| < \varepsilon_n \leq \frac{1}{n}$. for all $n \geq 1$

Thus $-\frac{1}{n} \leq a_n - \alpha \leq \frac{1}{n}$ for all $n \geq 1$

By Thm 1.8 (Squeeze Thm), $\lim\limits_{n\to\infty} (a_n - \alpha) = 0$

$\Rightarrow a_n = a_n - \alpha + \alpha$ is convergent & $\lim\limits_{n} a_n = \lim\limits_{n} (a_n - \alpha) + \alpha = \alpha$ $\square$

Definition 1.4: Let S' be the set of limit points of S. Then we define $\bar{S} := S' \cup S$, called the closure of S. If $S' \subset S$, or equivalently $S = \bar{S}$, then we say S is a closed set.

Examples of closed set of $\mathbb{R}$: $\mathbb{R}$, $[a, +\infty)$, $(-\infty, b]$ or $[a, b]$

Note (a,b), or $(a,b]$ are not closed as a is a limit pt of (a,b) & $a \notin (a,b)$.

Def 1.5: Let $\{I_n\}$ be a sequence of closed intervals such that

$$I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots$$

Let λ_n be the length of the interval I_n. We say $\{I_n\}$ is a sequence of closed nested intervals if it's in addition satisfies $\lambda_n \to 0$ as $n \to \infty$. In this case, we say $\{I_n\}$ form a nest.

Theorem 1.14: Let $\{I_n\}$ be a sequence of closed nested intervals. then there exists an unique ξ s.t.

$$\xi \in I_n \quad \text{for all } n \geq 1.$$

equivalently, $\bigcap_{n \geq 1} I_n = \{\xi\}$

Proof: We write $I_n = [a_n, b_n]$. Since $I_{n+1} \subset I_n$, it holds true that $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ which implies that $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$. for all $n \geq 1$.

Thus $\{a_n\}$ is a monotone increasing sequence that is bounded above (by any b_n). By Thm 1.10, there is a ξ s.t. $\lim_{n\to\infty} a_n = \xi$

Similarly $\{b_n\}$ is a monotone decreasing sequence that is bounded below. By corollary of Thm 1.10, $\exists\, \eta$ s.t.

$$\lim_{n\to\infty} b_n = \eta.$$

By Thm 1.9, $a_n \le b_n$ for all n implies $\xi \le \eta$. Since $a_n \le \xi \le \eta \le b_n$ for all $n \ge 1$, it holds true that

$$\xi \in I_n \text{ for all } n. \qquad (*)$$

To show ξ is the only point that $(*)$ holds true, we just need to show if ξ' satisfies $(*)$, then $\xi' = \xi$.

Now since $a_n \le \xi \le b_n$ ~~for all $n \ge 1$ (by $(*)$)~~

& $a_n \le \xi' \le b_n$,

we have $0 \le |\xi - \xi'| \le b_n - a_n = \lambda_n$ for all n.

But $\{I_n\}$ is a nest implies $\lambda_n \to 0$ as $n \to \infty$, by squeeze Thm, as a ~~conse~~ constant sequence

$$|\xi - \xi'| = \lim_{n\to\infty} |\xi - \xi'| = 0 \quad \Rightarrow \quad \xi = \xi' \qquad \square$$

Theorem 1.15 (Bolzano-Weierstrass Theorem [for sets])

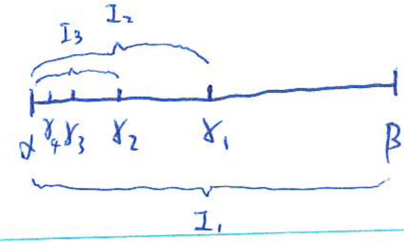
Every bounded, infinite set of real numbers has at least one limit point.

Proof: Let S be a bounded, infinite set. Then there are real numbers α & β s.t.

$$\alpha \le x \le \beta \quad \forall x \in S.$$

Let I_1 denotes $[\alpha, \beta]$. Denote by γ_1 the midpt $\frac{\alpha+\beta}{2}$ of I_1, and consider two closed intervals $[\alpha, \gamma_1]$ & $[\gamma_1, \beta]$. At least one of them must contain an infinite number of points in S. Denote that interval by I_2.

22



Then we may divide I_2 into two closed intervals, by introducing the midpoint γ_2 of I_2. Proceed by induction, we obtain a closed ~~nested~~ sequence of intervals $\{I_n\}_{n\geq 1}$ with the properties

① $I_n \supset I_{n+1}$, $\forall n \geq 1$

② $|I_n| = \cancel{\frac{\beta-\alpha}{2^{n+1}}}$ $\frac{\beta-\alpha}{2^{n-1}} \to 0$ as $n \to \infty$

③ Each I_n contains infinite number of pts in S.

① & ② ~~Thus~~ $\Rightarrow$ $\{I_n\}$ forms a nest. By Theorem 1.14

$$\bigcap_{n=1}^{\infty} I_n = \{\xi\} \quad \text{for some } \xi \in \mathbb{R}.$$

Claim: ξ is a limit pt of G.

Proof: $\forall \varepsilon > 0$, Let n be a large integer s.t.

$$\frac{\beta-\alpha}{2^{n-1}} < \varepsilon.$$

Since $\xi \in I_n$ & $|I_n| = \frac{\beta-\alpha}{2^{n-1}}$, we must have that

$$|x-\xi| < |I_n| = \frac{\beta-\alpha}{2^{n-1}} < \varepsilon \quad \forall x \in I_n.$$

Now I_n contains an infinite number of pts in G. So we may pick an $\bar{x} \in I_n \cap G$ & $\bar{x} \neq \xi$ which implies

$$|\bar{x}-\xi| < \varepsilon,$$

concluding the proof □

Theorem 1.16 [Bolzano-Weierstrass Theorem (for sequences)]
Every bounded sequence of real numbers has at least one convergent subsequence.

Proof: Let $\{a_n\}$ be a bounded sequences of real numbers. We divide the discussion into two different cases.

Case I: Suppose there is a real number ξ that appears in the sequence $\{a_n\}$ infinitely many times. Then it means there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ s.t.

$$a_{n_k} = \xi \qquad \forall k \geq 1.$$

Hence $\lim_{k\to\infty} a_{n_k} = \xi$ since it's a constant sequence.

Case II: Suppose we are not in case I. Then no number occurs infinitely many times in $\{a_n\}$. In particular the set $S = \{a_n, n\in\mathbb{Z}_+\}$ is infinite. Hence by Thm 1.15, S has a limit pt ξ. Then by definition of limit pt, for $\varepsilon_1 = 1$, there is a n_1 s.t.

$$|a_{n_1} - \xi| < \varepsilon_1 = 1$$

Then we may set $\varepsilon_2 = \min\{|a_n - \xi|,\ 1\leq n\leq n_1,\ \frac{1}{2}\}$ for which there is a n_2 s.t.

$$|a_{n_2} - \xi| < \varepsilon_2 < \frac{1}{2}$$

since $|a_{n_2} - \xi| < |a_n - \xi|$ for all $1\leq n\leq n_1$, we must have $n_2 > n_1$. Proceed by induction, suppose we've found $a_{n_1}, a_{n_2}, \dots, a_{n_k}$, then by setting

$$\varepsilon_{k+1} = \min\{|a_n - \xi|,\ 1\leq n\leq n_k,\ \tfrac{1}{k+1}\},$$

we can find a n_{k+1} s.t.

$$|a_{n_{k+1}} - \xi| < \varepsilon_{k+1} < \frac{1}{k+1}$$

Since $|a_{n_{k+1}} - \xi| < |a_n - \xi|$ for all $1\leq n\leq n_k$ we must have

$$n_{k+1} > n_k.$$

Thus we constructed a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ s.t.

$$|a_{n_k} - \xi| < \frac{1}{k} \implies \lim_{k\to\infty} a_{n_k} = \xi \qquad \square$$