Chapter 2. Continuous Functions

2.1 Def of continuous functions

Let $I \subset \mathbb{R}$ be an interval. So it could be closed, e.g $[a,b]$ or $[a,+\infty)$, open e.g. (a,b) or $(a,+\infty)$, or half-open, e.g $[a,b)$. Then a point $x_0 \in I$ is called an interior point of an interval I if x_0 is not an endpoint of I.
An open interval (a,b) containing a pt x_0 is called a neighborhood of x_0. The interval $(x_0-\delta, x_0+\delta)$ is called a δ-neighborhood of x_0.

Def 2.1: ① Let I be an interval & ~~x~~ $\xi \in I$ be an interior pt. Let $I' = I \setminus \{\xi\}$. Let $f(x)$ be a function defined on I or I'. Suppose there is a number A s.t. the following holds:

$$\left.\begin{aligned} &\forall \varepsilon>0,\ \exists \delta>0 \text{ s.t.} \\ &\qquad |f(x)-A|<\varepsilon, \text{ for all } 0<|x-\xi|<\delta \text{ and } x\in I \end{aligned}\right\} (6)$$

$$(\text{or } x \in (\xi-\delta, \xi+\delta)\cap I,\ x \neq \xi)$$

Then we say that $f(x)$ converges (or tends) to A as x converges (or tends) to ξ and write

$$\lim_{x\to\xi} f(x) = A, \quad \text{or} \quad f(x)\to A \text{ as } x\to\xi.$$

② Suppose (6) is replaced by

$$|f(x)-A|<\varepsilon \text{ for all } \xi<x<\xi+\delta,\ x\in I;$$

here ξ may be either an interior point of I or the left endpoint of I. We then say that $f(x)$ tends to A as x tends to ξ from the right, and write

$$\lim_{x\to\xi+} f(x) = A, \quad \lim_{x\to\xi+0} f(x) = A, \quad \lim_{x\downarrow\xi} f(x) = A$$

or $f(x)\to A$ as $x\downarrow\xi$, $f(\xi+)=A$, or $f(\xi+0)=A$

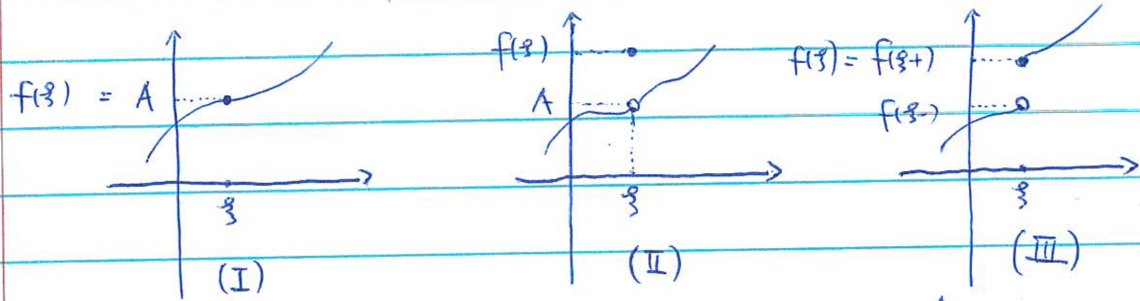
③ Similarly if (6) is replaced by

$$|f(x) - A| < \varepsilon, \text{ for all } \xi - \delta < x < \xi, \ x \in I,$$

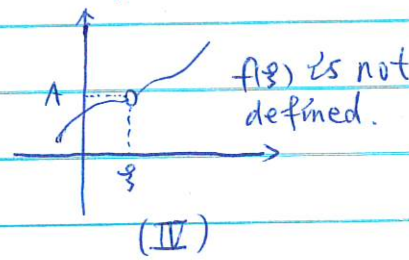
then ξ can ~~e.g.~~ either be an interior pt of I or the rightend point of I. In this case, we say $f(x)$ tends to A as x tends to ξ from the left & write

$$\lim_{x \to \xi -} f(x) = A, \quad \lim_{x \nearrow \xi} f(x) = A, \quad f(x) \to A \text{ as } x \nearrow \xi$$

$$f(\xi -) = A, \text{ or } f(\xi - 0) = A.$$



We are mostly interested in case (I) above, which is called continuous function (at ξ).



Def 2.2: Let $f: I \to \mathbb{R}$ & $\xi \in I$ ~~be an interior point~~. Then we say f is continuous at ξ if $\lim_{x \to \xi} f(x)$ exists & equals $f(\xi)$. If f is continuous at every pt $x \in I$, then we say f is continuous on I.

Three different cases here:

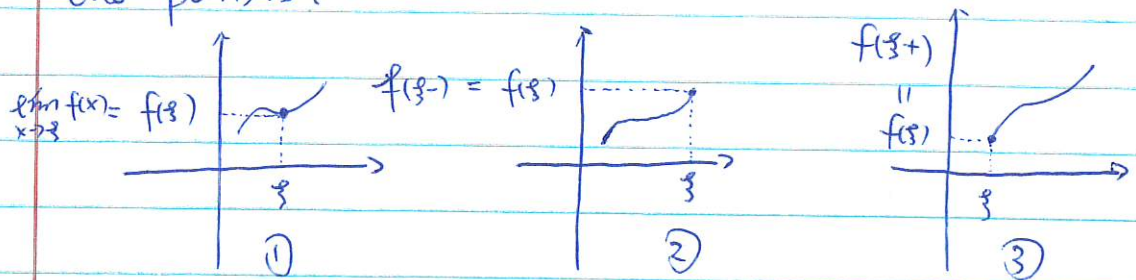
① If ξ is an interior pt of I, then Def 2.2 means

$$\lim_{x \to \xi} f(x) = f(\xi)$$

② If $\xi \in I$ & is the right end point of I, then Def 2.2 means $\lim_{x\to\xi-} f(x) = f(\xi)$, or $f(\xi-) = f(\xi)$

③ If $\xi \in I$ & is the left end point of I, then Def 2.2 means $\lim_{x\to\xi+} f(x) = f(\xi)$, or $f(\xi+) = f(\xi)$.

Note if I is an open interval, e.g. $I=(a,b)$, then continuity is only in the sense of ① as I contains no end-points.



If ξ is an interior pt of I & $f(\xi-)$ exists, then we say f is left continuous at ξ. If $f(\xi+)$ exists, then we say f is right continuous at ξ. Note by definition, f is continuous at ξ if & only if $f(\xi-) = f(\xi+) = f(\xi)$.

From now on, we will mostly focus on continuity case ①. Let's rewrite the definition of continuity of f at ξ. Recall $\lim_{x\to\xi} f(x) = A$ means: $\forall \varepsilon>0$, $\exists \delta>0$ s.t.

$$|f(x)-A| < \varepsilon, \quad \forall x \in (\xi-\delta, \xi+\delta)\cap I,\ x\neq\xi$$

Now in case of continuity, since $A = f(\xi)$, ~~so~~ $f(x) = f(\xi)$ when $x=\xi$. So we may rewrite the definition as

$\lim_{x \to \xi} f(x) = f(\xi)$ means: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$|f(x) - f(\xi)| < \varepsilon$, $\forall x \in (\xi - \delta, \xi + \delta) \cap I$.

(or $\forall |x - \xi| < \delta$, $x \in I$).

Again, geometrically f is continuous at ξ means as x tends to ξ, $f(x)$ tends to $f(\xi)$. Roughly speaking, no matter how small the $\varepsilon > 0$ is, if x is sufficiently close to ξ (like $|x - \xi| < \delta$), then $f(x)$ is close to $f(\xi)$ within distance ε, e.g. $|f(x) - f(\xi)| < \varepsilon$.

Examples: All elementary functions are continuous on the domain where they can be defined. What is a elementary function? It's a function of one variable which is the composition of a finite number of arithmetic operations $(+, -, \times, \div)$, exponentials, logarithms, constants, and solution of algebraic equations (a generalization of nth roots $(x)^{\frac{1}{n}}$). There are five main types:

① $f(x) = x^n$ powers of x
$f(x) = x^{\frac{1}{n}}$ roots of x $\Big\} \rightsquigarrow f(x) = x^a$, $a \in \mathbb{R}$ power function

② $f(x) = e^x$ or generally $f(x) = a^x$, $a > 0$, called exponential functions

③ $f(x) = \sin x$, $\cos x$, $\tan x$, ... called trigometric functions

④ Inverse function of exponential function:
$f(x) = \log(x)$, logarithmic function $\quad \log(e^x) = x$, $e^{\log x} = x$

⑤ Inverse function of trigonometric functions:

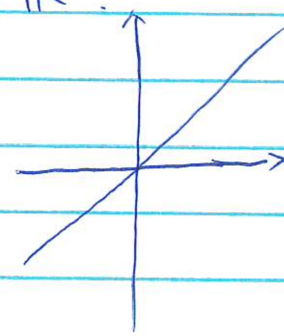
$f(x) = \arcsin(x)$ (or $\sin^{-1}(x)$),

$\arctan(x)$ (or $\tan^{-1}(x)$), ...

They are all continuous. But to prove it, it takes some work. In fact, we haven't ~~define~~ rigorous defined some of them, e.g. x^a when $a \notin \mathbb{Q}$, $\log(x)$, $\arctan(x)$.

We start with the simplest ones, i.e. $f(x) = x^n$.

Ex 1. First, we show $f(x) = x$ is continuous on $\mathbb{R}$.



Proof: To show $f(x)=x$ is continuous on $\mathbb{R}$, we just need to show $f(x)$ is continuous at every point $\xi \in \mathbb{R}$.

Fix any $\xi \in \mathbb{R}$, to show $f(x)=x$ is continuous at ξ, we need to show:

$\forall \varepsilon > 0$, $\exists \delta$, s.t.

$$|f(x) - f(\xi)| < \varepsilon, \quad \forall\ |x-\xi| < \delta$$ (no need to consider I here since $I = \mathbb{R}$)

Question: how to find the $\delta > 0$?

Work backwards: $|f(x) - f(\xi)| < \varepsilon$

$\Leftarrow |x-\xi| < \varepsilon$ since $f(x) = x$ $\Leftarrow$ we may take $\delta = \varepsilon$.

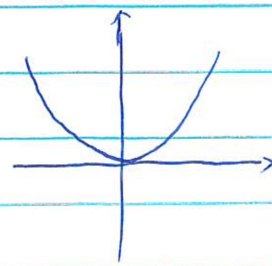
Then $\forall \varepsilon > 0$, set $\delta = \varepsilon > 0$, then

$$|f(x) - f(\xi)| = |x - \xi| < \varepsilon, \quad \forall\ |x-\xi| < \delta = \varepsilon,$$ done

(i.e. $|x-\xi| < \varepsilon \Rightarrow |x-\xi| = |f(x) - f(\xi)| < \varepsilon$) $\square$

Ex 2: $n=2$, i.e. show $f(x)=x^2$ is continuous on $\mathbb{R}$

Proof: Fix any $\xi \in \mathbb{R}$, we want to show
$\forall \varepsilon>0$, $\exists \delta>0$ s.t.
$$|f(x)-f(\xi)|<\varepsilon, \quad \forall\ |x-\xi|<\delta.$$



Again, how to find the δ?

Work backwards:
$$|f(\xi)-f(x)|<\varepsilon$$
$$\Leftrightarrow |x^2-\xi^2|<\varepsilon \Leftrightarrow |(x+\xi)(x-\xi)|<\varepsilon \Leftrightarrow |x+\xi|\cdot|x-\xi|<\varepsilon$$

Now since $|x-\xi|<\delta \Rightarrow |x|<|\xi|+\delta$, & $|x+\xi|\le|x|+|\xi|<2|\xi|+\delta$

$$\Leftarrow (2|\xi|+\delta)|x-\xi|<\varepsilon$$

First ~~control~~ condition on δ: $\delta\le 1$

$$\Leftarrow (2|\xi|+1)|x-\xi|<\varepsilon$$

$$\Leftarrow |x-\xi|<\frac{\varepsilon}{2|\xi|+1}$$

second condition on δ: $\delta \le \varepsilon/_{2|\xi|+1}$

Hence going backwards, we obtain:
$\forall \varepsilon>0$, set $\delta=\min\{1, \varepsilon/_{2|\xi|+1}\}$, then
$$|f(x)-f(\xi)|<\varepsilon, \quad \forall\ |x-\xi|<\delta.$$

Ex 3: $f(x)=x^n$ is continuous on $\mathbb{R}$.

Proof: Fix any $\xi\in\mathbb{R}$, we need to show
$\forall\varepsilon>0$, $\exists\delta>0$ s.t. $|f(x)-f(\xi)|<\varepsilon$, $\forall\ |x-\xi|<\delta$.

Again, we work backwards to determine δ.

Here we are going to use the following formula:

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

where do we get this formula: Recall

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x} \quad \text{if } x \neq 1$$

Thus $a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}$

$$= b^{n-1}\left[\left(\frac{a}{b}\right)^{n-1} + \left(\frac{a}{b}\right)^{n-2} + \cdots + 1\right]$$

$$= b^{n-1}\,\frac{1-\left(\frac{a}{b}\right)^n}{1-\frac{a}{b}} = \frac{b^{n-1} - \frac{a^n}{b}}{1-\frac{a}{b}} = \frac{b^n - a^n}{b-a} = \frac{a^n - b^n}{a-b}$$

$$\Rightarrow a^n - b^n = (a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})(a-b).$$

Now $|f(x) - f(\xi)| < \varepsilon \Leftrightarrow |x^n - \xi^n| < \varepsilon$

$$\Leftrightarrow |x^{n-1} + x^{n-2}\xi + \cdots + x\xi^{n-2} + \xi^{n-1}| \cdot |x-\xi| < \varepsilon$$

First, we set $\delta \leq 1$. Then again $|x-\xi| < 1 \Rightarrow |x| < |\xi| + 1$

$$\Rightarrow |x^{n-i-1}\xi^i| < (|\xi|+1)^{n-i-1}|\xi|^i \leq (|\xi|+1)^{n-1}, \quad \forall i = 0, \ldots, n-1$$

$$\Rightarrow |x^{n-1} + x^{n-2}\xi + \cdots + x\xi^{n-2} + \xi^{n-1}| \leq \sum_{i=0}^{n-1} |x^{n-1-i}\xi^i| \leq \sum_{i=0}^{n-1} (|\xi|+1)^{n-1} = n(|\xi|+1)^{n-1}$$

$$\Leftarrow n(|\xi|+1)^{n-1}|x-\xi| < \varepsilon \Leftrightarrow |x-\xi| < \frac{\varepsilon}{n(|\xi|+1)^{n-1}}$$

Second condition on δ: $\delta \leq \dfrac{\varepsilon}{n(|\xi|+1)^{n-1}}$.

By the argument above, we obtain:

$\forall \varepsilon > 0$, set $\delta = \min\left\{1, \frac{\varepsilon}{n(|\xi|+1)^{n-1}}\right\}$, then

$|f(x) - f(\xi)| < \varepsilon$, $\forall\ |x-\xi| < \delta$. $\square$

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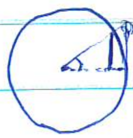
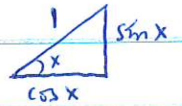
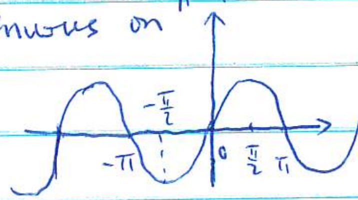
Ex 4: $f(x) = \sin(x)$ is continuous on $\mathbb{R}$







Fix any $\cancel{x}\ \xi \in \mathbb{R}$, we need to show that:

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - f(\xi)| < \varepsilon$, $\forall\ |x-\xi| < \delta$.

Here we are going to use:

① $\sin a - \sin b = 2 \sin\frac{a-b}{2} \cos\frac{a+b}{2}$

② $|\sin a| \le |a|$, $\forall a \in \mathbb{R}$.

$\cos a - \cos b$
$= 2 \sin\frac{a-b}{2} \sin\frac{a+b}{2}$

To $|f(\xi) - f(x)| < \varepsilon$

$\iff |\sin(x) - \sin(\xi)| < \varepsilon$

$\overset{①}{\iff} 2\left|\sin\frac{\overset{x}{\cancel{a}}-\xi}{2}\right| \cdot \left|\cos\frac{\cancel{a+b}\ x+\xi}{2}\right| < \varepsilon$

$\Leftarrow 2\left|\sin\frac{x-\xi}{2}\right| < \varepsilon$ since $|\cos x| \le 1$, $\forall x \in \mathbb{R}$.

$\overset{②}{\Leftarrow} 2\left|\frac{x-\xi}{2}\right| < \varepsilon \Leftarrow |x-\xi| < \varepsilon$

Thus we may take $\delta = \varepsilon$ as the process above shows $|x-\xi| < \varepsilon$ implies $|f(x) - f(\xi)| < \varepsilon$.

$\forall \varepsilon > 0$, set $\delta = \varepsilon$, then $|f(x) - f(\xi)| < \varepsilon$, $\forall\ |x-\xi| < \varepsilon$.

$\square$

Theorem 2.1. (Relations between convergent sequence & continuous functions)

Let $f: I \to \mathbb{R}$ be a function. Let $\xi \in I$. Then f is continuous at ξ if and only if the following holds:

Let $\{x_n\}$ be any sequence in I s.t. $\lim_{n\to\infty} x_n = \xi$. Then $\{f(x_n)\}$ is also convergent. Moreover $\lim_{n\to\infty} f(x_n) = f(\xi)$. $\quad (*)$

Proof: 1) "Only if" part, i.e. (continuity at $\xi \Rightarrow (*)$).

f is continuous at ξ: $\forall \varepsilon > 0$, $\exists \delta$, $|f(x) - f(\xi)| < \varepsilon$, $\forall\ |x - \xi| < \delta$, $x \in I$.

Now $\lim_{n\to\infty} x_n = \xi$ implies for $\delta > 0$, there exists a n_0 s.t.

$$|x_n - \xi| < \delta, \text{ if } n \geq n_0 \quad (\text{note } x_n \in I).$$

But $|x_n - \xi| < \delta$ & $x_n \in I$ implies (by $(*)$)

$$|f(x_n) - f(\xi)| < \varepsilon. \quad n \geq n_0$$

To summarize: $\forall \varepsilon > 0$, we found $n_0 \in \mathbb{Z}_+$ s.t.

$$|f(x_n) - f(\xi)| < \varepsilon, \quad \forall\ n \geq n_0$$

$\Rightarrow \lim_{n\to\infty} f(x_n)$ exists & equals $f(\xi)$.

2) "If" part, i.e. $(*) \Rightarrow$ continuity of f at ξ.

Argue by contradiction. Suppose f is not continuous at ξ. Let's see what we get from the definition. It implies there is a $\varepsilon_0 > 0$, s.t. for any $\delta > 0$,

$$|f(x) - f(\xi)| \geq \varepsilon_0 \quad \text{for some } x \text{ s.t. } |x - \xi| < \delta, \ x \in I$$

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Now taking $\delta = \frac{1}{n}$, we can find a x_n s.t.

$$\begin{cases} |x_n - \xi| < 1/n, \ x_1 \in I \\ |f(x_n) - f(\xi)| > \varepsilon_0 \end{cases} \Rightarrow \lim_{n\to\infty} x_n = \xi.$$

Thus we've found a sequence $\{x_n\}$ in I s.t.

$$\lim x_n = \xi$$

$$\& \quad f(x_n) \notin (f(\xi) - \varepsilon_0, \ f(\xi) + \varepsilon_0), \ \forall n \geq 1.$$

$$\Rightarrow f(x_n) \not\to f(\xi), \quad \text{as } n \to \infty.$$

This clearly contradicts with (*). So our assumption f is not continuous at ξ is false
$\Rightarrow f$ is continuous at ξ. □

Corollary: $\boxed{\text{Ex: } f(x) = \sin\frac{1}{x}}$ Similarly, $\lim_{x\to\xi} f(x) = A \iff$

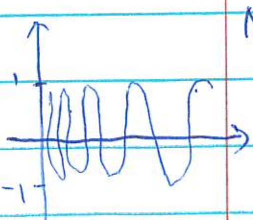
For any $\{x_n\}$ in $I \setminus \{\xi\}$ s.t. $\lim_{n\to\infty} x_n = \xi$, it holds that

$$\lim_{n\to\infty} f(x_n) = A.$$

Ex: $f(x) = \sin\frac{1}{x}$ on ~~$(-\infty, 0)$~~ $(0, +\infty)$ has no ~~left~~ right limit at 0.

Proof: Suppose $\lim_{x\to 0+} f(x) = A$. Then for any sequence of positive numbers $\{x_n\}$ with $\lim_{n\to\infty} x_n = 0$, we must have

$$\lim_{n\to\infty} f(x_n) = A.$$



Now let $x_n = \frac{1}{n\pi}$. Then $x_n \to 0$ & $f(x_n) = \sin\frac{1}{x_n} = \sin n\pi = 0$

$$\Rightarrow \lim_{n\to\infty} f(x_n) = 0 \quad ①$$

let $x_n = \frac{1}{2n\pi + \frac{\pi}{2}}$. Then $x_n \to 0$ & $f(x_n) = \sin\frac{1}{x_n} =$
$\sin(2n\pi + \frac{\pi}{2}) = \sin\frac{\pi}{2} = 1 \Rightarrow \lim_{n\to\infty} f(x_n) = 1 \quad ②$

$\Rightarrow f$ ~~0~~ has no limit at $0+$ since $\{f(x_n)\}$ is not convergent. □

Def 2.3: (limits at $\pm\infty$)

① Let $f: [a,+\infty) \to \mathbb{R}$ (or on $(a,+\infty)$). Suppose there is a $A \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists M > a$ s.t.

$$|f(x) - A| < \varepsilon, \quad \forall\, x > M.$$

Then we say $f(x)$ tends to A as x tends to ∞, denoted

$$\lim_{x\to\infty} f(x) = A$$

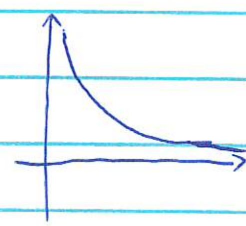
② Similarly, $f: (-\infty, b] \to \mathbb{R}$ (or on $(-\infty, b)$). Suppose there is a A s.t. $\forall \varepsilon > 0$, $\exists M < b$ s.t.

$$|f(x) - A| < \varepsilon, \quad \forall\, x < M.$$

Then we say $f(x)$ tends to A as x tends to $-\infty$, denoted

$$\lim_{x\to-\infty} f(x) = A.$$

Examples: ① $\lim\limits_{x\to\infty} \frac{1}{x} = 0$



Proof: $\forall \varepsilon > 0$, $\exists\, \cancel{\delta}\ M > 0 = ?$ s.t.

$|f(x) - 0| < \varepsilon, \quad \forall\, x > M.$

"Goal: need to find M s.t. $x > M \Rightarrow |f(x) - 0| < \varepsilon$"

work backwards:

$|f(x) - 0| < \varepsilon \iff \frac{1}{x} < \varepsilon \iff x > \frac{1}{\varepsilon}$

Hence $M = \frac{1}{\varepsilon}$. In summary,

$\forall \varepsilon > 0$, $\exists M = \frac{1}{\varepsilon}$, s.t. $|f(x) - 0| < \varepsilon$, $\forall\, x > M$.

② Similarly, one can show $\lim\limits_{x\to\infty} \frac{1}{x^k} = 0$, $\forall k \geq 1$

$\forall \varepsilon > 0$, $\exists M > 0$ s.t. $|f(x) - 0| < \varepsilon$, $\forall\, x > M$

$|f(x) - 0| = \frac{1}{x^k} < \varepsilon \iff \frac{1}{\varepsilon} < x^k \iff \sqrt[k]{\frac{1}{\varepsilon}} < x$, hence $M = \frac{1}{\sqrt[k]{\varepsilon}}$ or $\frac{1}{\varepsilon^{\frac{1}{k}}}$

Mimicking the proof of Thm 2.1, one can easily get

Corollary: $\lim_{x\to\infty} f(x) = A$ if & only if

$\lim_{n\to\infty} f(x_n) = A$ for all $\{x_n\} \subseteq (a, +\infty)$ s.t. $\lim_{n\to\infty} x_n = +\infty$.

In particular, $\lim_{x\to+\infty} \frac{1}{x^k} = 0$

implies $\lim_{n\to\infty} \frac{1}{n^k} = 0$, taking $x_n = n$ then $f(x_n) = \frac{1}{x_n^k} = \frac{1}{n^k}$

so $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{1}{n^k} = 0$

i.e. $\lim_{x\to\infty} \frac{1}{x^k} = 0$ is a generalization of $\lim_{n\to\infty} \frac{1}{n^k} = 0$.

Def 2.4: (convergence to infinity)

Let $\xi \in I$
$I' = I \setminus \{\xi\}$

① Let $f: I' \to \mathbb{R}$ ~~$\xi \in I$~~. Suppose ~~there is a A s.t.~~ it holds that

$\forall N > 0$, $\exists \delta > 0$ s.t. $f(x) > N$, $\forall 0 < |x - \xi| < \delta$, $x \in I$.

Then we say $f(x)$ tends to ∞ as x tends to ξ

$\lim_{x\to\xi} f(x) = +\infty$.

② Similarly one can define $\lim_{x\to\xi} f(x) = -\infty$.

Examples ① $\lim_{x\to 0} \frac{1}{x} = \infty$ view $f = \frac{1}{x}: (0, +\infty) \to \mathbb{R}$.

Proof: $\forall N > 0$, $\exists \delta = ?$ s.t. $f(x) > N$, $\forall 0 < x < \delta$

$f(x) > N \Leftrightarrow \frac{1}{x} > N \Leftrightarrow \frac{1}{N} > x$, $\delta = \frac{1}{N}$.

② If we view $f(x) = \frac{1}{x}: (-\infty, 0) \to \mathbb{R}$, then $\lim_{x\to 0} f(x) = -\infty$

$\forall N > 0$, $\exists \delta > 0 = ?$ s.t. $f(x) < -N$, $\forall -\delta < x < 0$

$f(x) < -N \Leftrightarrow \frac{1}{x} < -N \Leftrightarrow -\frac{1}{N} < x \Rightarrow \delta = \frac{1}{N}$. $\square$