2.2 Operations with continuous Functions

Let f & g be two functions defined on a set I. Then the functions $f+g$, $f-g$, $f\cdot g$, and f/g (if $g\neq 0$) are defined by

$$\left.\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (f-g)(x) &= f(x)-g(x)\\ (fg)(x) &= f(x)\cdot g(x)\\ (f/g)(x) &= f(x)/g(x)\end{aligned}\right\}\quad \text{for any } x\in I.$$

Theorem 2.2 (Sum of continuous functions are continuous)
Let $f, g : I \to \mathbb{R}$ be both continuous at $\xi \in I$. Then $f+g$ is continuous at ξ.

Proof: We have two different ways to prove it. The first is to prove by definition & the second is to use Theorem 2.1

- First proof:

$\lim_{x\to\xi} f(x) = f(\xi)$ means $\forall \varepsilon>0$, $\exists \delta_1$ s.t. $|f(x)-f(\xi)|<\frac{\varepsilon}{2}$, if $|x-\xi|<\delta_1$ & $x\in I$

$\lim_{x\to\xi} g(x) = g(\xi)$ means $\forall \varepsilon>0$, $\exists \delta_2$, s.t. $|g(x)-g(\xi)|<\frac{\varepsilon}{2}$, if $|x-\xi|<\delta_2$ & $x\in I$

Now we set $\delta=\min\{\delta_1,\delta_2\}$. Then $\forall \varepsilon$, & and for the choice of δ, we have if $|x-\xi|<\delta$ & $x\in I$, then

$$\begin{aligned}|(f+g)(x)-(f+g)(\xi)| &= |f(x)+g(x)-[f(\xi)+g(\xi)]|\\ &= |(f(x)-f(\xi))+(g(x)-g(\xi))|\\ &\leq |f(x)-f(\xi)|+|g(x)-g(\xi)|\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\ &= \varepsilon\end{aligned}$$

This is nothing other than $\lim_{x\to\xi}(f+g)(x) = (f+g)(\xi)$
i.e. $f+g$ is continuous at ξ.

- Second prove : Recall f is continuous at ξ

$$\Longleftrightarrow \lim_{n\to\infty} f(x_n) = f(\xi), \text{ if } \{x_n\} \text{ in } I \text{ satisfying } \lim_{n\to\infty} x_n = \xi.$$

Now f & g are continuous at ξ implies $\forall \{x_n\}$ in I with $\lim_{n\to\infty} x_n = \xi$, it holds that

$$\lim_{n\to\infty} f(x_n) = f(\xi) \quad \& \quad \lim_{n\to\infty} g(x_n) = g(\xi)$$

By "operations of limits", $\{f(x_n) + g(x_n)\} = \{(f+g)(x_n)\}$ is also convergent, moreover

$$\begin{aligned} \lim_{n\to\infty} (f+g)(x_n) &= \lim [f(x_n) + g(x_n)] \\ &= \lim f(x_n) + \lim g(x_n) \\ &= f(\xi) + g(\xi) \\ &= (f+g)(\xi) \end{aligned}$$

This implies $f+g$ is continuous at ξ.

Theorem 2.3 (Product of continuous functions are continuous)

~~Proof: Here we adopt the first proof (by def)~~

Let $f, g : I \to \mathbb{R}$ be continuous at ξ. The $f \cdot g$ is continuous at ξ.

Proof: Here we adopt the first proof by definition.

Again, $\lim_{x\to\xi} f(x) = f(\xi)$ means $\forall \varepsilon$, $\exists \delta_1$ s.t

$$|f(x) - f(\xi)| < \boxed{\tfrac{1}{2(|g(\xi)|+1)}} \cdot \varepsilon, \quad \text{if } |x-\xi| < \delta_1, x \in I.$$

$\lim_{x\to\xi} g(x) = g(\xi)$ means $\forall \varepsilon$, $\exists \delta_2$, s.t.

$$|g(x) - g(\xi)| < \boxed{\tfrac{1}{2(|f(\xi)|+1)}} \cdot \varepsilon, \quad \text{if } |x-\xi| < \delta_2, x \in I.$$

Here we need a little fact:

First, we pick δ_0 s.t. if $|x-\xi|<\delta_0$, $x\in I$, then

$$|f(x)-f(\xi)|<1 \quad (\text{i.e. } \delta_0 \text{ is the } \delta \text{ for } \varepsilon=1)$$

Then $|f(x)|-|f(\xi)|<1 \Rightarrow |f(x)|<|f(\xi)|+1$, $\forall\, |x-\xi|<\delta_0$, $x\in I$.

Now we set $\delta=\min\{\delta_0,\delta_1,\delta_2\}$, then if $|x-\xi|<\delta$, $x\in I$, we have

$$
\begin{aligned}
&|(f\cdot g)(x)-(f\cdot g)(\xi)|\\
&=|f(x)\cdot g(x)-f(\xi)\cdot g(\xi)|\\
&=|f(x)\cdot g(x)-f(x)\cdot g(\xi)+f(x)\cdot g(\xi)-f(\xi)\cdot g(\xi)|\\
&\le |f(x)\cdot g(x)-f(x)\cdot g(\xi)|+|f(x)\cdot g(\xi)-f(\xi)\cdot g(\xi)|\\
&=|f(x)|\cdot|g(x)-g(\xi)|+|g(\xi)|\cdot|f(x)-f(\xi)|\\
&<(|f(\xi)|+1)\cdot|g(x)-g(\xi)|+|g(\xi)|\cdot|f(x)-f(\xi)|\\
&<(|f(\xi)|+1)\frac{1}{2|f(\xi)|+1}\cdot\varepsilon+|g(\xi)|\frac{1}{2(|g(\xi)|+1)}\cdot\varepsilon\\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}
$$

$\square$

Theorem 2.4 (Quotient of continuous functions are continuous)
Let $f,g: I\to\mathbb{R}$ be continuous at $\xi\in I$ & $g(x)\neq 0$, $\forall x\in I$.
Then f/g is continuous at ξ.

Proof: Here we adopt the second proof.
f,g are continuous at $\xi \Rightarrow \forall \{x_n\}$ in I with $\lim x_n=\xi$, we have $\lim\limits_{n\to\infty} f(x_n)=f(\xi)$ & $\lim\limits_{n\to\infty} g(x_n)=g(\xi)$. By "operations of limits" $\frac{f(x_n)}{g(x_n)}$ is also convergent & $\lim\limits_{n\to\infty}\frac{f(x_n)}{g(x_n)}=\frac{\lim f(x_n)}{\lim g(x_n)}=\frac{f(\xi)}{g(\xi)}=\left(\frac{f}{g}\right)(\xi)$
$\Rightarrow f/g$ is continuous at ξ. $\square$

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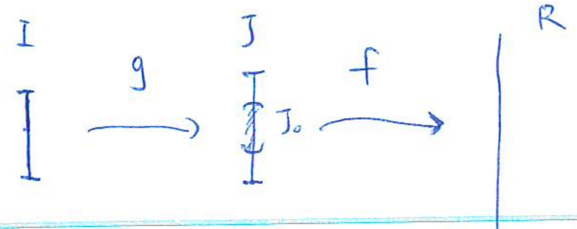


Definition 2.5 (composition of functions)

Let $g: I \to \mathbb{R}$ be a function. Let $J_0 = g(I) = \{g(x) : x \in I\}$, i.e. the range of g. Let J be another interval s.t. $J_0 \subset J$. Let $f: J \to \mathbb{R}$ be a function. Then for each $x \in I$, we have a number $f(g(x))$, denoted by $h(x)$. Thus, we obtained a new function $h(x) = f(g(x))$, sometimes we write

$$h = f \circ g,$$ called the composition of f ~~&~~ with g.

Theorem 2.5 (composition of continuous functions are continuous)

Let f & g be as in Def 2.5. Let g be continuous at $\xi \in I$ & f be continuous at $g(\xi) \in J$. Then $h = f \circ g$ is continuous at ξ.

Proof: ① f is continuous at $g(\xi)$ means:

$\forall \varepsilon > 0$, $\exists \eta > 0$, s.t. $|f(y) - f(g(\xi))| < \varepsilon$, if $|y - g(\xi)| < \eta$, $y \in J$

② g is continuous at ξ ~~means~~ implies

for the $\eta > 0$ above, $\exists \delta > 0$, s.t. $|g(x) - g(\xi)| < \eta$, if $|x - \xi| < \delta$, $x \in I$.

& obviously $g(x) \in J_0 \subset J$.

① & ② together implies:

$\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. if $|x - \xi| < \delta$, $x \in I$

then $|g(x) - g(\xi)| < \eta$ & $g(x) \in J$, which in turn implies

$$|f(g(x)) - f(g(\xi))| < \varepsilon.$$

To viz: $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. if $|x - \xi| < \delta$, then

$$|h(x) - h(\xi)| < \varepsilon$$

$\Rightarrow h$ is continuous at ξ. $\square$

Examples

① Recall we had a long proof that $f(x) = x^n$ is continuous. But it follows easily from the facts:

- $f(x) = x$ is continuous
- Theorem 2.3, i.e. product of continuous functions are continuous.

Because $x^n = \underbrace{x \cdot x \cdots x}_{n}$ is continuous since it's a finite product of n continuous $f(x) = x$.

② More generally, ~~any~~ polynomials $p(x) = \sum_{k=0}^{n} a_k \cdot x^k$ are continuous. Indeed for each k,

$a_k \cdot x^k$ is continuous as it's a product of constant a_k & x^k, both of which are continuous.

Now $p(x)$ is a finite sum of continuous functions, hence, itself is continuous.

③ Even more generally, any quotient of polynomials

$$r(x) = \frac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 + b_1 x + \cdots + b_m x^m}$$

is continuous at every pt where the denominator is not zero. Because it's a quotient of two polynomials, which are continuous. By Thm 2.4, $r(x)$ is continuous. Such functions are called rational functions.

(41)

(4) $\sin(x^2)$, $\sin^2(x)$, $\tan(x)$ are continuous.

Let $f(x)=x^2$ & $g(x)=\sin(x)$. Then both f & g are continuous. Hence by Theorem 2.5

$f\circ g(x)=f(g(x))$ is continuous, which is $g^2(x)=\sin^2 x$

& $g\circ f$ is continuous, which is $\sin(f(x))=\sin(x^2)$.

$\tan(x)=\frac{\sin x}{\cos x}$ are quotient of two continuous functions, hence is continuous at every pt it's well defined. In other words, at every pt wher $\cos(x)\neq 0$, $\tan(x)$ is continuous. ~~But~~

~~$\tan(x)$ is~~.

(5) Compute $\lim_{n\to\infty}\sin\left(\frac{1}{\sqrt{n+1}}\right)$

$\sin\left(\frac{1}{\sqrt{n+1}}\right)=\sin\sqrt{\frac{1}{n+1}}$ Note $h(x)=\sin\sqrt{x}$ is continuous on $[0,+\infty)$ as it's a composition of continuous functions.

Hence if $\lim_{n\to\infty}x_n=\xi$, then $\lim_{n\to\infty}h(x_n)=h(\xi)$

Clearly $\lim_{n\to\infty}\frac{1}{n+1}=0$, then

$$\lim_{n\to\infty}\sin\sqrt{\frac{1}{n+1}}=\lim_{n\to\infty}h\left(\frac{1}{n+1}\right)=h(0)=\sin\sqrt{0}=\sin 0=0$$

2.3 Maximum & Minimum

Definition 2.6: Let $f: I \to \mathbb{R}$ be a function. Again we let $f(I) = \{f(x) : x \in I\}$ be the range of f. If $f(I)$ is bounded above, then we say f is bounded above. This happens if & only if $\exists M \in \mathbb{R}$ s.t.

$$f(x) \le M, \quad \forall x \in I.$$

The supremum, i.e. the least upper bound, of $f(I)$ is called the supremum of f on I. We denote it by

$$\sup_{x \in I} f(x), \quad \text{or } \text{l.u.b.}\, f(x)$$

In other words $\alpha = \sup_{x \in I} f(x)$ if & only if

- $f(x) \le \alpha, \quad \forall x \in I$ (α is an upper bound of $f(I)$)
- $\forall \varepsilon > 0, \exists x \in I$, s.t. $f(x) > \alpha - \varepsilon$ (no number smaller than α can be an upperbound of $f(I)$)

Similarly, if $f(I)$ is bounded below, then we say f is bounded below. The infimum (or greatest lower bound) of $f(I)$ is called the infimum of f on I, ~~i.e. $\beta =$~~ We denote it by

$$\inf_{x \in I} f(x), \quad \text{or } \text{g.l.b.}\, f(x)$$

In other words, $\beta = \inf_{x \in I} f(x)$ if & only if

- $f(x) \ge \beta, \quad \forall x \in I$, ($\beta$ is a lower bound of $f(I)$)
- $\forall \varepsilon > 0, \exists x \in I$ s.t. $f(x) < \beta + \varepsilon$ (no number bigger than β can be a lower bound of $f(I)$)

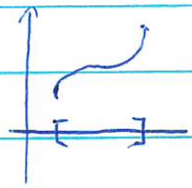
If $f(I)$ is bounded, then we say f is a bounded function on I. This happens if & only if $\exists M > 0$, s.t.

$$|f(x)| \le M, \quad \forall x \in I$$

Theorem 2.6: Let f be a continuous function on a closed, bounded interval $I = [a,b]$. Then f is bounded.

~~Proof: Argue by contradiction. Suppose f is not~~ bounded. Then no $M > 0$ can be satisfy

$$|f(x)| \leq M, \ \forall x \in I.$$

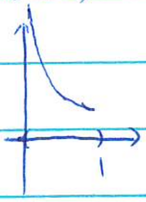


In particular, $\forall n \in \mathbb{Z}_+$, $\exists x_n \in I$ s.t.

$$|f(x_n)| > n. \qquad (*)$$

Since $x_n \in I, \forall n \in \mathbb{Z}_+$, $\{x_n\}$ is bounded sequence. By Bolzano-Weierstrass Theorem for sequences, there is subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that is convergent. In other words, $\exists \xi \in \mathbb{R}$ s.t. $\lim_{k\to\infty} x_{n_k} = \xi$. We first claim that $\xi \in I$.

Thm 2.6 is not true if I is not closed, e.g $f(x) = \frac{1}{x}$ on $(0,1)$



Indeed, either $\exists k_0 > 0$, s.t. $x_{n_{k_0}} = \xi$, $\boxed{\forall k \geq k_0}$ then $\xi = x_{n_{k_0}} \in I$; or $x_{n_k} \neq \xi$, $\forall k \geq 1$. But in this case $\lim_{k\to\infty} x_{n_k} = \xi$ implies $\forall \varepsilon > 0$, $\exists k_0 > 0$ s.t.

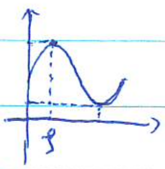
$$|x_{n_k} - \xi| < \varepsilon, \ \forall k \geq k_0 \quad \& \quad x_{n_k} \neq \xi, x_{n_k} \in I$$

By definition, ξ is a limit point of I. But $I = [a,b]$ is a closed interval which is a closed set, i.e. a set contains all its limit point. So $\xi \in I$.

Now by continuity of f & $\lim_{k\to\infty} x_{n_k} = \xi \in I$, we obtain $\lim_{k\to\infty} f(x_{n_k}) = f(\xi)$. In particular $\{f(x_{n_k})\}$ is a bounded sequence (since it's convergent). But by $(*)$,

$$|f(x_{n_k})| > n_k, \ \forall k \geq 1, \ \& \ n_k \to \infty \text{ as } k \to \infty$$

Hence $\{f(x_{n_k})\}$ is not bounded. This is a contradiction, which is from the false assumption that ~~for~~ f is not bounded on I. Thus f is bounded.



Definition 2.7: Let $f: I \to \mathbb{R}$ be bounded above. Suppose $\exists \xi \in I$ s.t. $f(x) \le f(\xi)$, $\forall x \in I$. Then we call ξ a maximum point of f and we call $f(\xi)$ the maximum of f on I. We also say that f has a maximum, and that it assumes (or attains) its maximum on I at ξ. Similgarly, we can define minimum pt & minimum of f.

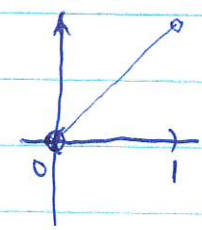
Remark: ① It's cleard that if $f(\xi)$ is the maximum of f on I, then $f(\xi) = \sup_{x \in I} f(x)$. Indeed,

(I). $f(x) \le f(\xi)$, $\forall x \in I$ by definition, i.e. $f(\xi)$ is an upper bound.

(II). $\forall \varepsilon > 0$, $f(\xi) > f(\xi) - \varepsilon$, i.e. no number smaller than $f(\xi)$ can be an upper bound of f on I.

Similarly, $\min_{x \in I} f(x) = \inf_{x \in I} f(x)$. (or minimum)

② f might not have maximum, even if f is continuous on I. Example: $f(x) = x$ on $(0,1)$. Clearly, $\sup_{x \in I} f(x) = 1$, but 1 cannot be attained by any pt in $(0,1)$. The problem here is that the end pt 1 of I is not in I, i.e. I is not closed. Actually, we have the following theorem.



Theorem 2.7: Let $f(x)$ be a continuous function on a closed, bounded interval $I = [a,b]$. Then f has maximum & minimum on I.

Proof: By Thm 2.6, f is bounded on I, i.e. $\exists M > 0$ s.t.

$$|f(x)| < M, \ \forall x \in I.$$

In particular, $f(I)$ has a supremum, say $\sup_{x \in I} f(x) = \alpha$.

By definition of supremum, $\forall \varepsilon > 0$, $\exists X_\varepsilon \in I$ s.t.

$$\alpha \geq f(X_\varepsilon) > \alpha - \varepsilon$$

In particular, we may pick $\varepsilon = \frac{1}{n}$ for each $n \geq 1$. Then for each $n \geq 1$, we obtain a $X_n \in I$ s.t.

$$\alpha - \frac{1}{n} < f(X_n) \leq \alpha, \quad \forall n \geq 1.$$

By squeeze Thm, $\{f(X_n)\}$ is convergent & $\lim_{n\to\infty} f(X_n) = \alpha$.

On the other hand, $\{X_n\}$ is in I which is bounded. By Bolzano-Weierstrass for sequences, $\exists$ ~~sy~~ subsequence $\{X_{n_k}\}$ of $\{X_n\}$ that is convergent, i.e. $\exists \xi \in \mathbb{R}$ s.t.

$$\lim_{k\to\infty} X_{n_k} = \xi.$$

Again by the same argument we used in the proof of Thm 2.6, $\xi \in I$. By continuity of f, & $\lim_{k\to\infty} X_{n_k} = \xi$, we must have $\lim_{k\to\infty} f(X_{n_k}) = f(\xi)$

But $\{f(X_{n_k})\}$ is a subsequence of the convergent sequence $\{f(X_n)\}$, hence

$$\underbrace{\lim_{k\to\infty} f(X_{n_k})}_{= f(\xi)} = \underbrace{\lim_{n\to\infty} f(X_n)}_{= \alpha}$$

$\Rightarrow \alpha = f(\xi)$, i.e. $f(\xi)$ is the maximum & ξ is ~~the~~ a maximum point of f on I. Indeed by definition of α: $f(x) \leq \alpha = f(\xi)$, $\forall x \in I$.

The proof of existence of minimum point is completely similar.

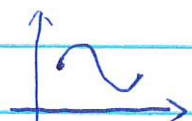
Basically f is bounded $\Rightarrow \exists \beta = \inf_{x\in I} f(x)$. By definition of infimum, ~~so~~ $\forall \varepsilon > 0$, $\exists Y_\varepsilon \in I$ s.t. $\beta \leq f(Y_\varepsilon) < \beta + \varepsilon$. Then one may proceed by choosing $\{Y_n\}$ s.t. $\beta \leq f(Y_n) < \beta + \frac{1}{n}$. $\square$

2.4. Intermediate Values

The main goal of section is to prove the following Intermediate Theorem

Theorem 2.8. Let $f: I \to \mathbb{R}$ be a continuous function defined on the closed, bounded interval $I = [a, b]$. Assume that $f(a) \neq f(b)$ and let γ be any number between $f(a)$ & $f(b)$. Then there exists at least one pt $\cancel{c \in [a,b]}$ $c \in (a, b)$ s.t. $f(c) = \gamma$.

Remark: ① Thm 2.8 basically says that ~~if~~ a continuous function f may not go from $f(a)$ to $f(b)$ without passing through any intermediate value bewteen $f(a)$ & $f(b)$. This matches the geometrical vision of continuity.



② In fact, we may take any two points x_1 & x_2 s.t. $a < x_1 < x_2 < b$, then $f(x)$ must attain any value between $f(x_1)$ & $f(x_2)$.

③ In particular, for the f as in Thm 2.8, by Thm 2.7, $\exists\, \xi_1$ & ξ_2 s.t. $f(\xi_1) = \min_{x \in I} f(x)$ & $f(\xi_2) = \max_{x \in I} f(x)$. Thus by Thm 2.8, any value between $f(\xi_1)$ & $f(\xi_2)$ must be attained by some x ~~beth~~ between ξ_1 & ξ_2, in particular $x \in I$. We thus proved the following corollary under the assumption that Thm 2.8 holds true:

Corollary: Let $f: I \to \mathbb{R}$ be continuous on $I = [a, b]$. Then $f(I) = [m, M]$ where m is the minumum of f on I & M is the maximum of f on I.

(47)

Proof of Thm 2.8: Without loss of generality, we assume $f(a) < f(b)$. Divide $[a,b]$ into two sub-intervals with equal length by introducing its mid-pt $c_1 = \frac{b+a}{2}$.

set $I_1 = I = [a,b]$

If $f(c_1) = \gamma$, then we are done. Otherwise:

(I) if $f(c_1) < \gamma$, then we set $I_2 = [c_1, b]$

(II) if $f(c_1) > \gamma$, then we set $I_2 = [a, c_1]$

$\Rightarrow$ $f(\text{left end pt}) < \gamma < f(\text{righ end pt})$

Now we repeat the same process on I_2 by introducing its mid-pt c_2, i.e. if $f(c_2) = \gamma$, then the process terminates as we are done, otherwise we find an interval I_3, the length of which is half of that of I_2 & $f(\text{left end pt}) < \gamma < f(\text{right end pt})$

By induction, either we ~~found~~ find at step n a c_n s.t. $f(c_n) = \gamma$; or we find a sequence of intervals

$$I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n \supset \cdots$$

with the following properties: set $I_n = [a_n, b_n]$, then

- $a_n \le a_{n+1} \le b_{n+1} \le b_n$, $\forall n \ge 1$
- $b_n - a_n = \frac{b-a}{2^{n-1}}$
- $f(a_n) < \gamma < f(b_n)$, $\forall n \ge 1$

$\Rightarrow \exists \xi$ s.t. $\{\xi\} = \bigcap_{n\ge 1} I_n$ as $\{I_n\}$ form a nest

i.e. $\lim_{n\to\infty} a_n = \xi = \lim_{n\to\infty} b_n$

By continuity of $f \Rightarrow \lim_{n\to\infty} f(a_n) = f(\xi) = \lim_{n\to\infty} f(b_n)$

But $\forall n$ $f(a_n) < \gamma \Rightarrow \lim f(a_n) \le \gamma$, i.e. $f(\xi) \le \gamma$

$\forall n$, $f(b_n) > \gamma \Rightarrow \lim f(b_n) \ge \gamma$, i.e. $f(\xi) \ge \gamma$

$\Rightarrow$ $f(\xi) = \gamma$, we are done

$\square$