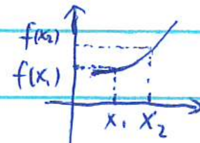
2.5 Monotone Functions & Inverse Functions

Definition 2.8: Let $f: I \to \mathbb{R}$ be a function. We say f is

(I) monotone increasing on I if:

$$f(x_1) \le f(x_2), \quad \forall\ x_1 < x_2 \in I.$$

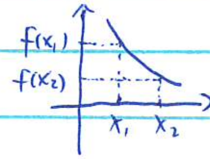


(II) strictly monotone increasing on I if:

$$f(x_1) < f(x_2), \quad \forall\ x_1 < x_2 \in I.$$

(III) monotone decreasing on I if:

$$f(x_1) \ge f(x_2), \quad \forall\ x_1 < x_2 \in I.$$

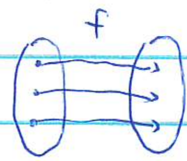


(IV) strictly monotone decreasing on I if:

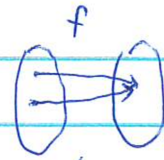
$$f(x_1) > f(x_2), \quad \forall\ x_1 < x_2 \in I.$$

A (strictly) monotone function is a function that is either (strictly) monotone increasing or (strictly) monotone decreasing

Definition 2.9: Consider a function $f: I \to \mathbb{R}$. We say f is one-to-one if $f(x_1) \ne f(x_2)$ $\forall\ x_1 \ne x_2$.



one-to-one



not one-to-one

Let $J = f(I)$. Then for each $y \in J$, there is an unique $x \in I$ s.t. $f(x) = y$.

Thus we may define a new function

$$g: J \to I \quad \text{as} \quad g(y) = x, \text{ where } f(x) = y. \qquad (*)$$

Such function g is called the inverse of $f: I \to J$, denoted $g = f^{-1}$.

Clearly, by $(*)$, $g(f(x)) = x$, $f(g(y)) = y$, i.e.

$g \circ f : I \to I$ is the identity on I

$f \circ g : J \to J$ is the identity on J.

Remark: Comparing Def 2.8 & Def 2.9, if $f: I \to \mathbb{R}$ is strictly monotone on I, then $f(x_1) \ne f(x_2)$ $\forall\ x_1 \ne x_2 \in I$.

Thus there exists the inverse $g: J \to I$ of f, $J = f(I)$.
Moreover, if f is increasing, then so is g. Indeed, if $y_1 < y_2$, then $g(y_1) < g(y_2)$. Because otherwise if $g(y_1) > g(y_2)$, we have $f(g(y_1)) > f(g(y_2))$, i.e. $y_1 > y_2$ contradicts $y_1 < y_2$.
Similarly, if f is decreasing, then so is g.

Theorem 2.9: Let f be a strictly monotone, continuous function on a closed, bounded interval $I = [a, b]$. Then its inverse g as a function defined on $J = f(I)$ is strictly monotone & continuous.

Proof: Without loss of generality, we may assume that f is increasing. By the Remark before Thm 2.9,

$$g: [f(a), f(b)] \to \mathbb{R}, \quad g = f^{-1}$$

exists & is strictly monotone increasing. Thus we only need to show that g is continuous on $[f(a), f(b)]$, i.e. we need to show that g is continuous at any $\gamma \in [f(a), f(b)]$.

We focus on the case $\gamma \in (f(a), f(b))$. The cases of end points can be discussed similarly.

In this case, clearly there is a $\xi \in (a, b)$ s.t.

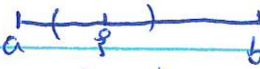
$$g(\gamma) = \xi, \quad \text{or } f(\xi) = \gamma$$

To show g is continuous at γ, by definition, $\forall \varepsilon > 0$, there is a $\delta > 0$ s.t. $|g(y) - g(\gamma)| < \varepsilon$, $\forall\ |y - \gamma| < \delta$, $y \in J$.

Actually, to show continuity, we only need to consider sufficiently small ε as $|g(y) - g(\gamma)| < \varepsilon_1 \Rightarrow |g(y) - g(\gamma)| < \varepsilon_2$ if $\varepsilon_2 > \varepsilon_1$.

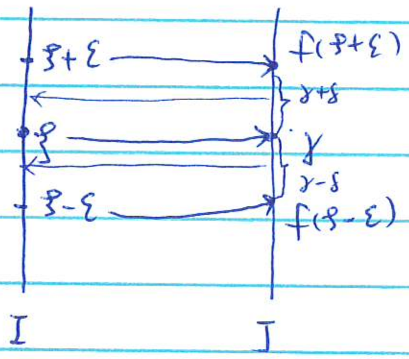
Thus we may fix any sufficiently small $\varepsilon > 0$ s.t.

$$[\xi-\varepsilon, \xi+\varepsilon] \subset [a,b]$$



For example, we may choose $\varepsilon < \min\{\xi - a, b - \xi\}$.

By the fact that f is strictly monotone increasing, it holds that



$$f(\xi-\varepsilon) < f(\xi) < f(\xi+\varepsilon)$$

Set $\delta = \min\{f(\xi) - f(\xi-\varepsilon), f(\xi+\varepsilon) - f(\xi)\} > 0$.

Then $f(\xi-\varepsilon) \le \gamma - \delta < \gamma + \delta \le f(\xi+\varepsilon)$ $(*)$

Now if $|y-\gamma| < \delta$, i.e. $\gamma - \delta < y < \gamma + \delta$ $(**)$

then by the fact that g is strictly monotone increasing & facts $(*)$ & $(**)$, it holds that

$$g(f(\xi-\varepsilon)) \le g(\gamma-\delta) < g(y) < g(\gamma+\delta) \le g(f(\xi+\varepsilon))$$

$\Rightarrow$ $\xi - \varepsilon < g(y) < \xi + \varepsilon$, recall $\xi = g(\gamma)$

$\Rightarrow$ $g(\gamma) - \varepsilon < g(y) < g(\gamma) + \varepsilon$

$\Rightarrow$ $|g(y) - g(\gamma)| < \varepsilon$.

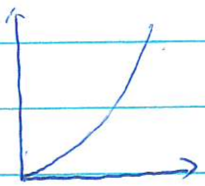
In summary, $\forall \varepsilon > 0$ small, $\exists \delta > 0$, s.t.

$$|g(y) - g(\gamma)| < \varepsilon, \text{ if } |y - \gamma| < \delta.$$

$\Rightarrow$ g is continuous at γ $\Rightarrow$ g is continuous on J as γ is arbitrarily chosen. □

Question: where did we use the fact f is continuous on $[a,b]$ in the proof of Theorem 2.9?

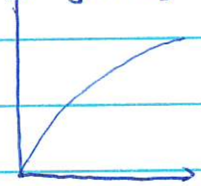
$f(x)=x^2$



Examples: ① Consider $f(x)=x^2 : [0,+\infty) \to [0,+\infty)$.
It's clearly that f is strictly monotone increasing on $[0,+\infty)$ & $f[0,+\infty) = [0,+\infty)$. Moreover, we showed that f is continuous. Thus, there is the inverse $g = f^{-1} : [0,+\infty) \to [0,+\infty)$ that is strictly monone increasing. In fact, we know

$$g(y) = \sqrt{y}$$

$g(y)=\sqrt{y}$



Moreover, we've show g is continuous by definition. But we may also deduce the continuity of g via Thm 2.9.

Note Thm 2.9 applies to f defined on a bounded interval $I=[a,b]$. To apply it to g, we just need to use the fact that continuity is a local statement. Precisely, $\forall \gamma \in [0,+\infty)$, we may ~~pick~~ set $\xi \in [0,+\infty)$

s.t. $f(\xi) = \gamma$, or $\xi = g(\gamma)$

Then we may pick a subinterval $I=[a,b] \ni \xi$ & hence $J=f(I)$ contains γ. Then f is strictly monotone increasing & continuous on $I \Rightarrow g$ is continuous on $J \ni \gamma$, in particular, g is continuous at γ. Since γ is arbitrary, we obtain g is continuous on $[0,+\infty)$.

② More generally, $\forall n \in \mathbb{Z}_+$, $f(x)=x^n : [0,+\infty) \to [0,+\infty)$ is strictly monotone increasing & continuous & $f[0,+\infty) = [0,+\infty)$.
Thus the inverse $g = f^{-1} : [0,+\infty) \to [0,+\infty)$ is strictly monotone increasing & continuous. In fact, $g(y) = y^{\frac{1}{n}}$.

③ $f(x) = \sin(x) : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is strictly monotone increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ & $f([-\frac{\pi}{2}, \frac{\pi}{2}]) = [-1, 1]$. Thus $g(y) = f^{-1}(y) = \arcsin(y)$ or $\sin^{-1}(y) : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is strictly monotone increasing & continuous. Similarly, one can deal with $\cos(x) : [0, \pi] \to [1, -1]$

④ $f(x) = \tan(x) : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is continuous & strictly monotone increasing & $f(-\frac{\pi}{2}, \frac{\pi}{2}) = \mathbb{R}$. Then $g(y) = \arctan(y)$ or $g(y) = \tan^{-1}(y) : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is strictly monotone increasing & continuous.

Exponentials & Logarithms.

Finally, we want to deal with exponential functions such as $f(x) = a^x$, $a > 0$.

However, so far, we only know the rigorous definition of a^r when $r \in \mathbb{Q}$, i.e. $r = \frac{m}{n}$, $m, n \in \mathbb{Z}$, $n \neq 0$.

Indeed, $a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m$ or $(a^m)^{\frac{1}{n}}$ for $\frac{m}{n} > 0$, where $a^{\frac{1}{n}}$ is defined as in example 2 for $n \geq 1$. If $\frac{m}{n} < 0$, then $a^{\frac{m}{n}} = \frac{1}{a^{-\frac{m}{n}}}$. If $\frac{m}{n} = 0$ then, $a^0 = 1$. In summary,

$$a^{\frac{m}{n}} = \begin{cases} (a^m)^{\frac{1}{n}} \text{ or } (a^{\frac{1}{n}})^m, & \frac{m}{n} > 0 \\ 1, & \frac{m}{n} = 0 \\ \frac{1}{a^{-\frac{m}{n}}} \text{ or } (\frac{1}{a})^{-\frac{m}{n}} & \frac{m}{n} < 0. \end{cases}$$

Moreover, one readily check (by induction for instance) that

$$a^m \cdot a^n = a^{m+n} \qquad (a^m)^n = a^{m \cdot n}, \quad \forall m, n \in \mathbb{Z}. \qquad (*)$$

One can easily extend $(*)$ to all $\alpha, \beta \in \mathbb{Q}$, i.e.

$$a^{\alpha} \cdot a^{\beta} = a^{\alpha+\beta}, \quad (a^{\alpha})^{\beta} = a^{\alpha \cdot \beta}, \quad \forall \alpha, \beta \in \mathbb{Q}. \qquad (\divideontimes)$$

Indeed, we may let $\alpha = \frac{p}{q}$, $\beta = \frac{m}{n}$. Then

(I) $a^{\alpha} \cdot a^{\beta} = a^{\frac{p}{q}} \cdot a^{\frac{m}{n}} = a^{\frac{pn}{qn}} \cdot a^{\frac{mq}{qn}} = (a^{\frac{1}{qn}})^{pn} \cdot (a^{\frac{1}{qn}})^{mq}$

by $(*)$, we obtain it equals $(a^{\frac{1}{qn}})^{pn+mq} \xlongequal{\text{definition}} a^{\frac{pn+mq}{qn}}$

$= a^{\frac{pn}{qn} + \frac{mq}{qn}} = a^{\frac{p}{q} + \frac{m}{n}} = a^{\alpha+\beta}$.

(II) For $(a^{\alpha})^{\beta} = a^{\alpha \cdot \beta}$, we first consider $\alpha = \frac{1}{q}$ & $\beta = \frac{1}{n}$.

Then $(a^{\alpha})^{\beta} = (a^{\frac{1}{q}})^{\frac{1}{n}} \Rightarrow [(a^{\alpha})^{\beta}]^{nq} = [[(a^{\frac{1}{q}})^{\frac{1}{n}}]^{n}]^{q} = (a^{\frac{1}{q}})^{q} = a$

But $(a^{\alpha\beta})^{nq} = (a^{\frac{1}{nq}})^{nq} = a \Rightarrow (a^{\alpha})^{\beta} = a^{\alpha\beta} \qquad (**)$

as $x^{nq} = a$ the solution is unique on $[0, +\infty)$.

Now for general $\alpha = \frac{p}{q}$ & $\beta = \frac{m}{n}$, we have

$(a^{\alpha})^{\beta} = (a^{\frac{p}{q}})^{\frac{m}{n}} \xlongequal{\text{def}} (((a^{p})^{\frac{1}{q}})^{\frac{1}{n}})^{m} \xlongequal{(**)} ((a^{p})^{\frac{1}{qn}})^{m}$

$\xlongequal{\text{def}} (a^{\frac{p}{qn}})^{m} \xlongequal{\text{def}} ((a^{\frac{1}{qn}})^{p})^{m} \xlongequal{(*)} (a^{\frac{1}{qn}})^{pm} \xlongequal{\text{def}} a^{\frac{pm}{qn}} = a^{\frac{p}{q} \cdot \frac{m}{n}}$

$= a^{\alpha \cdot \beta}$.

Now we are ~~so~~ almost ready to define a^{x} for $x \notin \mathbb{Q}$.

We still need ~~one~~ two more facts: ① for $a > 1$, the function $f(r) = a^{r}$ as a function on $\mathbb{Q}$ is strictly monotone increasing.

Proof: First, we show $a^{\frac{1}{n}} > 1$. Indeed, if $a^{\frac{1}{n}} \leq 1$, then $(a^{\frac{1}{n}})^{n} \leq 1^{n} \Rightarrow a' \leq 1$ contradicts with $a > 1$.

Consequently, $\forall r = \frac{m}{n} > 0$, it holds that

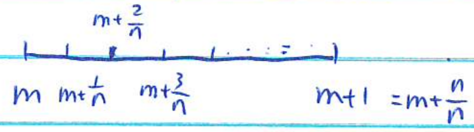
$a^{r} = a^{\frac{m}{n}} = (a^{m})^{\frac{1}{n}} > 1$ since $a^{m} > 1$ when $a > 1$.

Now if $r>s \in \mathbb{Q}$, then $r-s>0$ & $r-s \in \mathbb{Q}$. Thus

$$a^{r-s}>1 \Rightarrow a^r > a^s, \quad \forall r>s \in \mathbb{Q}. \qquad \square$$

② Between any two real numbers, there is ~~an~~ a rational number.

Proof: Let $\alpha<\beta$ be any two real numbers. If there is an integer m s.t. $\alpha<m<\beta$, then we are done as $m \in \mathbb{Z} \subseteq \mathbb{Q}$. Otherwise $\exists\, m$ s.t. $m \le \alpha < \beta \le m+1$. ~~Then~~ let $\varepsilon = \beta-\alpha>0$. Then we can find a large n s.t. $\frac{1}{n} < \beta-\alpha$. Now we divide the interval $[m, m+1]$ into n small intervals with equal length $\frac{1}{n}$



Since $\frac{1}{n} < \beta-\alpha$, there must exist a j, ~~0~~ $1 \le j \le n-1$ s.t. $\alpha < m+\frac{j}{n} < \beta$. Because otherwise there is a j_0 $\in[0,n)$ s.t.

$$m+\frac{j_0}{n} \le \alpha < \beta \le m+\frac{j_0+1}{n}$$

which implies $\beta-\alpha < (m+\frac{j_0+1}{n}) - (m+\frac{j_0}{n}) = \frac{1}{n}$ contradicts with $\beta-\alpha>\frac{1}{n}$. Thus $m+\frac{j}{n}$ is the rational number we're looking for.

By ②, for any $x \in \mathbb{R}$, we can find a sequence of rational numbers r_n that is decreasing in n & $\lim_{n\to\infty} r_n = x$, or simply, $r_n \downarrow x$. Indeed, we may pick r_1 ~~s.t.~~ $\in \mathbb{Q}$ s.t.

$$x < r_1 < x+1$$

Then suppose $r_1, \dots, r_k$ are ~~p~~ given, we may pick $r_{k+1} \in \mathbb{Q}$ s.t.

$$x < r_{k+1} < \min\{x+\tfrac{1}{k+1}, r_k\},$$

$\Rightarrow r_k < r_{k+1}, \ \forall k \geq 1$ & $x < r_k < \frac{1}{k} + x, \ \forall k \geq 1$

$\Rightarrow r_k \downarrow x$ as $k \to \infty$.

Now for each $x \notin \mathbb{Q}$, we pick a sequence of rational numbers $r_n \downarrow x$. Then ~~we def~~ we claim

~~$\{a^{r_n}\}$ is a decreasing sequence bounded below~~

By fact ①, $\{a^{r_n}\}$ is decreasing & r_n is decreasing. $a^{r_n} > a^s$ for any $n \geq 1$, if we pick $s \in \mathbb{Q}$ to be s.t. $s < x$. Because then

$r_n > x > s, \ \forall n \geq 1$.

By fact 1, $a^{r_n} > a^s, \ \forall n \geq 1$. Thus $\{a^{r_n}\}$ is bounded below & monotone decreasing. Consequently

$\lim_{n\to\infty} a^{r_n}$ exists.

We define $a^x = \lim_{n\to\infty} a^{r_n}$ (**)

However, this definition might be problematic if a^x depends on the choice of the sequence $\{r_n\}$ as there are infinitely many such sequences $r_n \downarrow x$, $r_n \in \mathbb{Q}$.

Claim: (**) is independent of the choice of $r_n \downarrow x$, $r_n \in \mathbb{Q}$.

Proof: Let $\{r_n\}$ & $\{s_n\}$ be two such sequences, ~~to~~

Then $\forall n \geq 1$, $r_m < s_n$ for all m sufficiently large as r_m is decreasing to x & $s_n > x, \ \forall n \geq 1$.

$\Rightarrow$ ~~$\lim_{m\to\infty}$~~ $a^{r_m} < a^{s_n}, \ \forall n \geq 1$, & $\forall m$ large

This implies that $\lim_{m\to\infty} a^{r_m} < a^{s_n} \quad \forall n \geq 1$

$\Rightarrow \lim_{m\to\infty} a^{r_m} \leq \lim_{n\to\infty} a^{s_n}$

Switching the role of r_m & s_n, we obtain

$$\lim_{n\to\infty} a^{s_n} \leq \lim_{n\to\infty} a^{r_m}$$

$\Rightarrow \lim_{m\to\infty} a^{r_m} = \lim_{n\to\infty} a^{s_n}$ for any two sequences $r_m \downarrow x$, $s_n \downarrow x$, $s_n, r_m \in \mathbb{Q}$.

Thus (~~$\ast\ast$~~) is well-defined, i.e.

$$a^x := \lim_{n\to\infty} a^{r_n} \text{ is a solid definition.}$$

Thus we defined a function: For any $a > 1$,

$$f(x) = a^x : \mathbb{R} \to \mathbb{R}$$

Next, we explore the properties of f.

(I). f is strictly monotone increasing. We've already showed it when f is on $\mathbb{Q}$. Now $\forall x_1 < x_2 \in \mathbb{R}$, by fact ②, we can find two rational numbers r, s. s.t.

$$x_1 < r < s < x_2$$

Now let $r_n \downarrow x$ & $s_n \downarrow x_2$, then for all n large

$$x_1 < r_n < r < s < x_2 < s_n$$

$\Rightarrow a^{x_1} = \lim_n a^{r_n} < a^r < a^s < \lim_{n\to\infty} a^{s_n} = a^{x_2}$

$\Rightarrow a^{x_1} < a^{x_2}$. $\forall x_1 \leq x_2 \in \mathbb{R} \Rightarrow f(x) = a^x$ is strictly monotone increasing.

Clearly, $a^n \to \infty$ as $n\to\infty$ & $a^{-n} = (\frac{1}{a})^n \to 0$, as $n\to\infty$

$\Rightarrow f(\mathbb{R}) \in (0,+\infty)$

(II). f is continuous. First, we show the relation

$$a^\alpha \cdot a^\beta = a^{\alpha+\beta}$$ can be extended from $\mathbb{Q}$ to $\mathbb{R}$.

Indeed, $\forall x, y \in \mathbb{R}\backslash\mathbb{Q}$, we may pick two sequences in $\mathbb{Q}$ r_n & s_n s.t. $r_n \downarrow x$ & $s_n \downarrow y$. Then $r_n + s_n \downarrow x+y$, $r_n + s_n \in \mathbb{Q}$.

Thus: $a^x \cdot a^y = \lim_n a^{r_n} \lim_n a^{s_n} = \lim_n a^{r_n} \cdot a^{s_n}$

$$= \lim_n a^{r_n+s_n} = a^{x+y}$$

If only one of x, y is irrational, say x is irrational, then let $r_n \downarrow x$. ~~We obtain~~ Then $r_n + y \downarrow x+y$, & $r_n + y \in \mathbb{Q}$.

Thus: $a^x \cdot a^y = (\lim_{n\to\infty} a^{r_n}) \cdot a^y = \lim_{n\to\infty} a^{r_n} \cdot a^y = \lim_{n\to\infty} a^{r_n+y}$

$$= a^{x+y}.$$

Hence, we have $a^{\alpha+\beta} = a^\alpha \cdot a^\beta$, $\forall \alpha, \beta \in \mathbb{R}$.

Now to show $f(x) = a^x$ is continuous, we only need to show $f(x)$ is continuous at $x=0$. Indeed, if $\lim_{x\to 0} a^x = a^0 = 1$, then, $\forall \xi \in \mathbb{R}$, we have

$$\lim_{x\to\xi} a^x = \lim_{x-\xi\to 0} a^\xi \cdot a^{x-\xi} = a^\xi \lim_{x-\xi\to 0} a^{x-\xi} = a^\xi.$$

To show $\lim_{x\to 0} a^x = 1$, we ~~fix~~ pick $\forall \varepsilon > 0$, a $\delta = \frac{1}{n_0}$, where n_0 is so large that $1-\varepsilon < a^{-\frac{1}{n_0}} < a^{\frac{1}{n_0}} < 1+\varepsilon$. Then reason, we can pick such a n_0 is that

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1, \quad \forall a > 0$$

Indeed, if $a>1$, then $\{a^{\frac{1}{n}}\}$ is a monotone decreasing & bounded below by 1. Thus

$$\alpha = \lim_{n\to\infty} a^{\frac{1}{n}} \geq 1.$$

On the other hand, if $\beta>1$, then $\beta^n \to \infty$, as $n\to\infty$.

Thus for n large $\beta^n > a \Rightarrow \beta > a^{\frac{1}{n}} \Rightarrow \beta > \lim_n a^{\frac{1}{n}} = \alpha$

$\Rightarrow \alpha < \beta,\ \forall \beta > 1 \Rightarrow \alpha \leq 1.$

$$\Rightarrow \alpha = 1.$$

Similarly, we can show $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$, for $0<a<1$.

Thus, $\forall \varepsilon>0$, $\exists\, n_0$, s.t. $\forall n \geq n_0$ $\quad 1+\varepsilon > a^{\frac{1}{n}} > 1.$

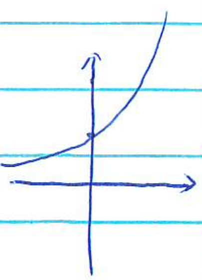
$\qquad\qquad\qquad\qquad\qquad\qquad \& \ 1-\varepsilon < a^{-\frac{1}{n}} = (\frac{1}{a})^{\frac{1}{n}} < 1$

$\Rightarrow 1-\varepsilon < a^{-\frac{1}{n_0}} < a^{\frac{1}{n_0}} < 1+\varepsilon.$

Now ~~set~~ $\forall$ x s.t. $|x|<\delta$, i.e. $-\overset{\frac{1}{n_0}\,\|}{\delta} < x < \overset{\|\,\frac{1}{n_0}}{\delta}$, we have by

(I) $\quad 1-\varepsilon < a^{-\frac{1}{n_0}} < a^x < a^{\frac{1}{n_0}} < 1-\varepsilon \Rightarrow |a^x - 1| < \varepsilon$

$\Rightarrow \lim_{x\to 0} a^x = 1 = a^0.$



Thus, for any $a>1$ $f(x)=a^x$ ~~is~~ $: \mathbb{R} \to (0,+\infty)$ is strictly monotone increasing & continuous since $\lim_{n\to\infty} f(n) = +\infty$, $\lim_{n\to-\infty} f(n) = 0$, we have $f(\mathbb{R}) = (0,+\infty)$.

If $a=1$, then $f(x) = a^x = 1^x \equiv 1$, $\forall x \in \mathbb{R}$.

If $a<1$, then we may define

$f(x) = a^x = (\frac{1}{a})^{-x}$ or $\dfrac{1}{(\frac{1}{a})^x}$, then it's clearly

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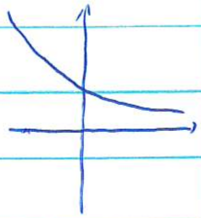
(*) In fact, we haven't showed the continuity of $f(x)=x^a$, when $a \notin \mathbb{Q}$. But actually, we may define $x^a := e^{a\log(x)}$ which is a composition of continuous functions, hence is continuous



that f is continuous & strictly monotone decreasing.

This takes care of exponential functions.

for $a>0$, $a \neq 1$

Now, since $f_a: \mathbb{R} \to (0,+\infty)$, $f_a(x) = a^x$ is strictly monotone & continuous, there exists

$$g_a: (0,+\infty) \to \mathbb{R}$$

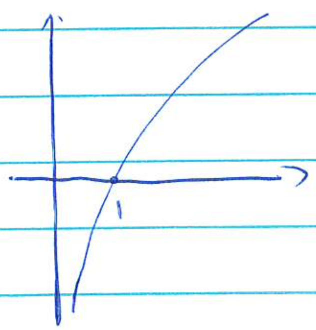
s.t. $g_a = f_a^{-1}$ that is strictly monotone & continuous.

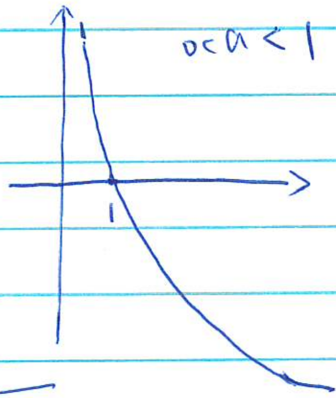
We denote such g_a as $g_a(x) = \log_a(x)$, i.e.

$$a^{\log_a x} = x, \ \forall x>0 \quad \&$$

$$\log_a a^x = x, \ \forall x \in \mathbb{R}.$$

is called the logarithmic functions.





As promised, we've showed the continuity of the basic elementary functions $f(x) = x^a$ (*), $\sin(x)$, a^x, $\arcsin(x)$, $\log_a x$.

By operations with continuous functions, all elementary functions are continuous whenever they are defined.