Math 150A, Winter 2020–Midterm Exam

This exam has 5 problems worth 50 points distributed over 7 pages, including this one and one blank page.

Instructions: This is a 70 minutes exam. You may not consult any notes or books during the exam, and no calculators are allowed. Show all of your work on each problem and carefully justify all answers. Points will be deducted for irrelevant, incoherent or incorrect statements, and no points will be awarded for illegible work. If you run out of room, you may work answers on the blank page. Be sure to clearly indicate when work is continued on another page.

NAME (please print):

NetID (please print):

	Question	Points	Score
Ryan —	1	8	
Jonathan —	2	12	
Ryan —	3	8	
Z.Z. —	4	10	
Ryan —	5	12	
	Total:	50	

1. (a) (3 points) State the definition of *convergent sequence.*

Let $\{a_n\}$ be a sequence & $L \in \mathbb{R}$. 1pt - setup

We say $\{a_n\}$ converges to L, if:

$\forall \varepsilon > 0, \exists n_0 \geq 1$ s.t. $|a_n - L| < \varepsilon \quad \forall n \geq n_0$ 2pts

(b) (5 points) Let $\{a_n\}$ be an eventually constant sequence. In other words, there is a $N \in \mathbb{Z}_+$ and $c \in \mathbb{R}$ such that $a_n = c$ for all $n \geq N$. Show by definition $\lim_{n\to\infty} a_n$ exists and equals c.

2pts

$\forall \varepsilon > 0$, we choose $n_0 = N$, then

$$\left.\begin{aligned} &|a_n - c| = |c - c| = 0 < \varepsilon \\ &\forall n \geq N. \end{aligned}\right\} \text{3pts}$$

By definition, $\lim_{n\to\infty} a_n = c$.

2. (a) (6 points) Show by definition that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

$\forall \varepsilon > 0$, we want to find $n_0 \geq 1$ s.t. $\forall n \geq n_0$ we have $\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$. 1pt

$$\left|\frac{1}{\sqrt{n}} - 0\right| = \frac{1}{\sqrt{n}} < \varepsilon \iff \frac{1}{n} < \varepsilon^2 \iff \frac{1}{\varepsilon^2} < n$$ 3pts

Thus taking $n_0 = \left\lceil \frac{1}{\varepsilon^2} \right\rceil$ (1pt), we have $\forall n \geq n_0$ $\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$ (1pt) 2pts

(b) (6 points) Determine the convergence of the following sequence

$$\left\{ a_n = \frac{(n+1)^2}{n} \right\}_{n \geq 1}.$$

If convergent, compute its limit; if divergent, explain your reasoning.

- We write $a_n = \frac{(n+1)^2}{n} = \frac{n^2+2n+1}{n} = n + 2 + \frac{1}{n}$ 1pt

- We know that $\{2+\frac{1}{n}\}_{n\geq 1}$ converges. Indeed $\lim_{n\to\infty} (2+\frac{1}{n}) = 2 + \lim \frac{1}{n} = 2$ 1pt

- We know that $\{n\}_{n\geq 1}$ is divergent since it's unbound 2pts

- Thus as a sum of a convergent sequence & a divergent sequence, $\{a_n\}$ is divergent. 2pts

3. (8 points) Use the fact that $e = \lim_{n\to\infty}(1+\frac{1}{n})^n$ to compute the limit of the sequence

$$\left\{a_n = \left(1+\frac{1}{n+2}\right)^n\right\}$$

$$- \quad a_n = \left(1+\frac{1}{n+2}\right)^n = \frac{\left(1+\frac{1}{n+2}\right)^{n+2}}{\left(1+\frac{1}{n+2}\right)^2}$$

2pts

$$- \quad \lim_{n\to\infty}\left(1+\frac{1}{n+2}\right)^2 = \left[\lim\left(1+\frac{1}{n+2}\right)\right]^2$$

$$= \left[1+\lim\frac{1}{n+2}\right]^2$$

$$= 1 \neq 0$$

1pt

$$- \quad \lim_{n\to\infty}\left(1+\frac{1}{n+2}\right)^{n+2} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

1pt

Since $\left\{\left(1+\frac{1}{n+2}\right)^{n+2}\right\}_{n\geq 1}$ is a subsequence of $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n\geq 1}$.

2pts

Thus, $\lim_{n\to\infty} a_n = \dfrac{\lim\left(1+\frac{1}{n+2}\right)^{n+2}}{\lim\left(1+\frac{1}{n+2}\right)^2} = \dfrac{e}{1} = e$ 2pts

4. Consider a sequence $a_1 = 3$ and $a_{n+1} = \sqrt{2 + a_n}$ for each $n \geq 1$.

(a) (6 points) Show by induction that $\{a_n\}$ is monotone decreasing and bounded below by 2.

- $\{a_n\}$ is monotone decreasing by induction:

$n=1$: $a_1 = 3 \geq \sqrt{2+3} = \sqrt{5} = a_2$ 1pt

Assume at $n=k$, $a_k \geq a_{k+1}$. Then at $n=k+1$:

$a_k \geq a_{k+1} \Rightarrow a_k + 2 \geq a_{k+1} + 2 \Rightarrow \sqrt{a_k+2} \geq \sqrt{a_{k+1}+2}$ 2pts

$\Rightarrow a_{k+1} \geq a_{k+2}$ $\Rightarrow a_n \geq a_{n+1}$ $\forall n \geq 1$.

- Bounded below by 2 via induction:

$n=1$: $a_1 \geq 2$. 1pt Assume $a_k \geq 2$. Then

$a_k + 2 \geq 2 + 2 \Rightarrow \sqrt{a_k+2} \geq \sqrt{4} = 2 \Rightarrow a_{k+1} \geq 2$ $\Rightarrow a_n \geq 2$ $\forall n \geq 1$ 2pts

(b) (4 points) From part (a), we know that $\lim_{n\to\infty} a_n$ exists. Compute it.

$\lim_{n\to\infty} a_n$ exists since it's bound below & monotone decreasing. $L \geq 2$ since it's the greatest lower bound (1pt)

Assume $L = \lim a_n$. Then $\lim a_{n+1} = L$

$\Rightarrow \lim_{n\to\infty} \sqrt{2+a_n} = \lim_{n\to\infty} a_{n+1} = L$ (1pt)

$\Rightarrow \left(\lim_{n\to\infty} \sqrt{2+a_n}\right) \cdot \left(\lim_{n\to\infty} \sqrt{2+a_n}\right) = L^2$

$= \lim_{n\to\infty} (\sqrt{2+a_n})^2 = \lim (2+a_n)$ 1pt

$= 2 + \lim a_n = 2 + L$

$\Rightarrow L^2 = 2 + L \Rightarrow L = -1$ or $L = 2$ 1pt

$\Rightarrow L = 2$

5. (a) (3 points) State the definition of *limit point.*

Let $S \subseteq \mathbb{R}$ and $\alpha \in \mathbb{R}$. (1pt)

Then we say α is a limit pt of S if: $\forall \varepsilon > 0$, $\exists x \in S \setminus \{\alpha\}$ s.t. $|x - \alpha| < \varepsilon$ (2pts)

(b) (3 points) State the definitions of *closure* and of *closed set.*

Let $S' = \{\alpha \in \mathbb{R} : \alpha \text{ is a limit pt of } S\}$ (1pt)

Then the closure of S is $\bar{S} = S \cup S'$ (1pt)

S is closed if $S = \bar{S}$ (or $S' \subseteq S$) (1pt)

(c) (6 points) Show that $S = \{-\frac{1}{n}, n \in \mathbb{Z}_+\}$ is not a closed set.

We have $\lim_{n \to \infty} (-\frac{1}{n}) = -\lim_{n \to \infty} \frac{1}{n} = 0$ 1pt

On the other hand $-\frac{1}{n} \in S$. $\forall n \geq 1$. 1pt

And $\{-\frac{1}{n}\}_{n \geq 1}$ is a sequence of mutually distinct pts 1pt

Thus 0 must be a limit pt of S or $0 \in S'$ 1pt

But $0 \notin S$. $\Rightarrow S' \nsubseteq S \Rightarrow S$ is not a closed set (2pts)