MATH 151B discussion, section 002 Ryan Ta
University of California, Riverside Winter 2021

Higher-Stakes Homework 1 solutions

1. Given the sequence of functions

$$f_n : [0, 1] \to \mathbb{R}$$
$$f_n(x) = \frac{x}{1 + nx}.$$

(1) Find the pointwise limit of the sequence and prove this limit.

Answer. Let $\epsilon > 0$ be given. Choose $N > \frac{1}{\epsilon}$. If $0 < x \leq 1$, then $n \geq N$ implies

$$\begin{aligned} |f_n(x) - 0| &= \left| \frac{x}{1 + nx} - 0 \right| \\ &= \frac{x}{1 + nx} \\ &< \frac{x}{nx} \\ &= \frac{1}{n} \\ &\leq \frac{1}{N} \\ &< \epsilon. \end{aligned}$$

If $x = 0$, then we have

$$\begin{aligned} |f_n(x) - 0| &= \left| \frac{0}{1 + n(0)} - 0 \right| \\ &= 0 \\ &< \epsilon. \end{aligned}$$

Therefore, $\{f_n\}$ converges to f. □

(2) Is the convergence uniform? Why?

Answer. The convergence is uniform because our choice of N depends only on ϵ (and not on x). □

2. Let the functions of functions $\{h_n\}$ defined by

$$h_n : [0, 1) \to \mathbb{R}$$
$$h_n(x) = \frac{x^n}{1 + x^{2n}}.$$

Does $\{h_n\}$ converge uniformly on $[0, 1)$? Why?

Answer. We will prove that $\{h_n\}$ converges pointwise to 0 on $[0, 1)$. (Compute for yourself that the limit is 0 just to be sure!) Let $\epsilon > 0$ be given. Choose $N > \log_x(\frac{1}{\epsilon})$. If $n \geq N$, then we have

$$\begin{aligned} |h_n(x) - 0| &= \left| \frac{x^n}{1 + x^{2n}} - 0 \right| \\ &= \frac{x^n}{1 + x^{2n}} \\ &< \frac{x^n}{1} \\ &= x^n \\ &\leq x^N \\ &< \epsilon \end{aligned}$$

for all $0 \leq x < 1$. Therefore, $\{h_n\}$ converges pointwise to 0 on $[0, 1)$.

Next, we will show that the convergence is not uniform. Choose for instance $\epsilon := \frac{1}{4}$. Then we have the uniform norm

$$\begin{aligned}
\|h_n - 0\|_u &= \sup_{x\in[0,1)} |h_n(x) - 0| \\
&= \sup_{x\in[0,1)} \left|\frac{x^n}{1+x^{2n}} - 0\right| \\
&= \sup_{x\in[0,1)} \frac{x^n}{1+x^{2n}} \\
&= \frac{1^n}{1+1^{2n}} \\
&= \frac{1}{2} \\
&> \frac{1}{4} \\
&= \epsilon.
\end{aligned}$$

So $\{h_n\}$ does not converge uniformly on $[0, 1)$. □

Alternate answer. To show that the convergence us not uniform on $[0, 1)$, we will employ a sequence argument. Choose for instance $\epsilon := \frac{1}{4}$ and $x_n := (1 - \frac{1}{2n})^{\frac{1}{n}}$. Then $\{x_n\}$ is a sequence contained in $[0, 1)$, and we would have

$$\begin{aligned}
|h_n(x_n) - 0| &= \left|\frac{x_n^n}{1+x_n^{2n}} - 0\right| \\
&= \frac{1-\frac{1}{2n}}{1+(1-\frac{1}{2n})^2} \\
&= \frac{1-\frac{1}{2n}}{1+(1-\frac{1}{n}+\frac{1}{4n^2})} \\
&= \frac{1-\frac{1}{2n}}{2-\frac{1}{n}+\frac{1}{4n^2}} \\
&= \frac{1-\frac{1}{2n}}{2-\frac{1}{n}+\frac{1}{4n^2}}\frac{4n^2}{4n^2} \\
&= \frac{2n(2n-1)}{8n^2-4n+1} \\
&> \frac{2n(2n-1)}{8n^2} \\
&= \frac{2n-1}{4n} \\
&= \frac{1}{2} - \frac{1}{4n} \\
&\geq \frac{1}{2} - \frac{1}{4(1)} \\
&= \frac{1}{4} \\
&= \epsilon.
\end{aligned}$$

Therefore, $\{h_n\}$ converges pointwise but not uniformly to 0 on $[0, 1)$. □

3. Find a sequence of continuous functions, $f_n : [0, 1] \to \mathbb{R}$ such that

$$\lim_{x\to 1}\left(\lim_{n\to\infty} f_n(x)\right) \quad \text{and} \quad \lim_{n\to\infty}\left(\lim_{x\to 1} f_n(x)\right) \quad \text{exist and}$$

$$\lim_{x\to 1}\left(\lim_{n\to\infty} f_n(x)\right) \neq \lim_{n\to\infty}\left(\lim_{x\to 1} f_n(x)\right).$$

Answer. Choose for instance $f_n(x) := x^n$, which is continuous for all $0 \leq x \leq 1$ and for all positive integers n. Then we would have

$$\begin{aligned}
&\lim_{x\to 1} f_n(x) = 1, \\
&\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1, \end{cases}
\end{aligned}$$

which imply

$$\begin{aligned}\lim_{n\to\infty}\lim_{x\to 1} f_n(x) &= \lim_{n\to\infty} 1\\ &= 1\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 1}\lim_{n\to\infty} f_n(x) &= \lim_{x\to 1^-} 0\\ &= 0.\end{aligned}$$

Therefore, the limits $\lim_{n\to\infty}\lim_{x\to 1} f_n(x)$ and $\lim_{x\to 1}\lim_{n\to\infty} f_n(x)$ exist and

$$\lim_{n\to\infty}\lim_{x\to 1} f_n(x) \neq \lim_{x\to 1}\lim_{n\to\infty} f_n(x),$$

as desired. □

4. Let $f_n : [0,\infty) \to \mathbb{R}$ be sequence of functions that are Riemann integrable on any interval of the form $[0,x]$, $x > 0$. Assume that f_n converges uniformly to a function f on $[0,\infty)$. Define the "average" sequence of functions:

$$\begin{aligned}&g_n : (0,\infty) \to \mathbb{R}\\ &g_n(x) = \frac{1}{x}\int_0^x f_n(t)dt\end{aligned}$$

and the function

$$\begin{aligned}&g : (0,\infty) \to \mathbb{R}\\ &g(x) = \frac{1}{x}\int_0^x f(t)dt\end{aligned}$$

Prove that g_n converges uniformly to g on $(0,\infty)$.

Answer. Let $\epsilon > 0$ be given. Since $\{f_n\}$ converges uniformly to f, there exists $N > 0$ such that, if $n \geq N$, then

$$|f_n(t) - f(t)| < \epsilon$$

for all $0 \leq t \leq x$. So, using the triangle inequality for integrals, we have

$$\begin{aligned}|g_n(x) - g(x)| &= \left|\frac{1}{x}\int_0^x f_n(t)\,dt - \frac{1}{x}\int_0^x f(t)\,dt\right|\\ &= \left|\frac{1}{x}\int_0^x f_n(t) - f(t)\,dt\right|\\ &\leq \frac{1}{x}\int_0^x |f_n(t) - f(t)|\,dt\\ &< \frac{1}{x}\int_0^x \epsilon\,dt\\ &= \frac{1}{x}(x-0)\\ &= \epsilon.\end{aligned}$$

Therefore, $\{g_n\}$ converges uniformly to g. □