Higher-Stakes Homework 2 solutions

1. Let $C([a,b]) = \{f : [a,b] \to \mathbb{R}, f \text{ is continuous}\}$. Show that

$$d(f,g) = \int_a^b \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\,dx$$

is a metric on $C([a,b])$.

Proof. Let $f, g, h \in X$ be given. For nonnegativity, we have

$$\begin{aligned} d(f,g) &= \int_a^b \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\,dx \\ &\geq \int_a^b \frac{0}{1+|f(x)-g(x)|}\,dx \\ &= 0 \end{aligned}$$

and that $d(f,g) = 0$ if and only if f, g satisfy

$$\int_a^b \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\,dx = 0,$$

if and only if

$$\frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} = 0$$

since the integrand is nonnegative (otherwise, a positive integrand in $[a,b]$ results in a positive integral over $[a,b]$), if and only if $|f(x)-g(x)| = 0$ for all $x \in [a,b]$, if and only $f(x) = g(x)$ for all $x \in [a,b]$, if and only if $f = g$. For symmetry, we have

$$\begin{aligned} d(f,g) &= \int_a^b \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\,dx \\ &= \int_a^b \frac{|g(x)-f(x)|}{1+|g(x)-f(x)|}\,dx \\ &= d(g,f). \end{aligned}$$

For triangle inequality, define $F : \mathbb{R} \to \mathbb{R}$ by $F(t) = \frac{t}{1+t}$. Then we have $F'(t) = \frac{1}{(1+t)^2} \geq 0$, meaning that f is increasing; that is, $a \leq b$ implies $F(a) \leq F(b)$ for any $a, b \in \mathbb{R}$. In particular, since the classical triangle inequality gives

$$|f(x)-g(x)| \leq |f(x)-h(x)| + |h(x)-g(x)|,$$

we have

$$F(|f(x)-g(x)|) \leq F(|f(x)-h(x)| + |h(x)-g(x)|)$$

So we have

$$\begin{aligned} d(f,h) &= \int_a^b \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\,dx \\ &= \int_a^b F(|f(x)-g(x)|)\,dx \\ &\leq \int_a^b F(|f(x)-h(x)| + |h(x)-g(x)|)\,dx \\ &= \int_a^b \frac{|f(x)-h(x)| + |h(x)-g(x)|}{1+|f(x)-h(x)| + |h(x)-g(x)|}\,dx \\ &= \int_a^b \frac{|f(x)-h(x)|}{1+|f(x)-h(x)| + |h(x)-g(x)|} + \frac{|h(x)-g(x)|}{1+|f(x)-h(x)| + |h(x)-g(x)|}\,dx \\ &= \int_a^b \frac{|f(x)-h(x)|}{1+|f(x)-h(x)| + |h(x)-g(x)|} + \int_a^b \frac{|h(x)-g(x)|}{1+|f(x)-h(x)| + |h(x)-g(x)|}\,dx \\ &\leq \int_a^b \frac{|f(x)-h(x)|}{1+|f(x)-h(x)|} + \int_a^b \frac{|h(x)-g(x)|}{1+|h(x)-g(x)|}\,dx \\ &= d(f,h) + d(h,g). \end{aligned}$$

Therefore, d is a metric, and so (X, d) is a metric space. □

2. Consider the sequences $\{f_n\}$ and $\{f_n'\}$, where $f_n(x) = \frac{1}{n}\exp(-n^2x^2)$ on the interval $[-1, 1]$. Show whether $\{f_n\}$ converges pointwise, uniformly, or if it diverges at some point. Do the same with $\{f_n'\}$. Justify your answers.

Proof. First, we claim that $\{f_n\}$ converges uniformly to 0 on $[-1, 1]$. Let $\epsilon > 0$ be arbitrarily small, and choose $N > \frac{\exp(-1)}{\epsilon}$. For all $n \geq N$, then we have $-n \leq -N$, and since $\exp(-x^2)$ is a decreasing function of x, we have $\exp(-n^2) \leq \exp(-N^2)$. Therefore, we obtain

$$\begin{aligned}
|f_n(x) - f(x)| &= \left|\frac{1}{n}\exp(-n^2x^2) - 0\right| \\
&= \frac{1}{n}\exp(-n^2x^2) \\
&\leq \frac{1}{n}\max_{x\in[-1,1]}\exp(-n^2x^2) \\
&= \frac{1}{n}\exp(-n^2(-1)^2) \\
&= \frac{1}{n}\exp(-n^2) \\
&\leq \frac{1}{n}\exp(-1^2) \\
&= \frac{\exp(-1)}{n} \\
&\leq \frac{\exp(-1)}{N} \\
&< \epsilon.
\end{aligned}$$

Therefore, $\{f_n\}$ converges uniformly to 0 on $[-1, 1]$.

Next, we claim that $\{f_n'\}$ converges pointwise to 0 on $[-1, 1]$. To this end, first we compute the first derivative of f_n, which is

$$\begin{aligned}
f_n'(x) &= \frac{d}{dx}\left(\frac{1}{n}\exp(-n^2x^2)\right) \\
&= \frac{1}{n}\exp(-n^2x^2)\frac{d}{dx}(-n^2x^2) \\
&= \frac{1}{n}\exp(-n^2x^2)(-2n^2x) \\
&= -2nx\exp(-n^2x^2)
\end{aligned}$$

Now, let $\epsilon > 0$ be given. First suppose $x \in [-1, 1] \setminus \{0\}$, and choose $N > \frac{2}{\epsilon x^2}$. First, notice that $\exp(x) \geq x$ implies in particular $\exp(-x^2) \leq \frac{1}{x^2}$ for all $x \in \mathbb{R}$. Therefore, for all $n \geq N$, we have

$$\begin{aligned}
|f_n'(x) - 0| &= |-2nx\exp(-n^2x^2) - 0| \\
&= 2n|x|\exp(-n^2x^2) \\
&\leq 2n(1)\exp(-n^2x^2) \\
&= 2n\exp(-n^2x^2) \\
&\leq 2n\frac{1}{n^2x^2} \\
&= \frac{2}{nx^2} \\
&\leq \frac{2}{Nx^2} \\
&< \epsilon.
\end{aligned}$$

Also, at $x = 0$, regardless of our choice of $N \in \mathbb{N}$, we have

$$\begin{aligned}
|f_n'(0) - 0| &= \left|-2n(0)\exp(-n^2(0)^2) - 0\right| \\
&= |0 - 0| \\
&= 0 \\
&< \epsilon.
\end{aligned}$$

Therefore, $\{f_n'\}$ converges pointwise to 0 on $[-1, 1]$. Now we will establish that the convergence is not uniform. Choose

$\epsilon := \exp(-1)$ and define the sequence $\{x_n\}$ defined by $x_n = \frac{1}{n}$. Then we have

$$\begin{aligned} |f_n'(x_n) - 0| &= |-2nx_n \exp(-n^2x_n^2) - 0| \\ &= 2n\frac{1}{n} \exp\left(-n^2\left(\frac{1}{n}\right)^2\right) \\ &= 2\exp(-1) \\ &> \exp(-1) \\ &= \epsilon. \end{aligned}$$

Therefore, $\{f_n'\}$ does not converge uniformly to 0 on $[-1, 1]$. □

3. For what values of $a > 0$ is the power series $\sum_{n=1}^{\infty} e^{-2n}x^n$ uniformly convergent in the interval $[0, a]$?

Proof. Notice that we can rewrite the power series as

$$\sum_{n=1}^{\infty} e^{-2n}x^n = \sum_{n=1}^{\infty}(e^{-2}x)^n,$$

which is a geometric series that converges provided that we impose

$$-1 < e^{-2}x < 1,$$

which is equivalent to

$$-e^2 < x < e^2.$$

In other words, the radius of convergence is $R := e^2$. By Proposition 6.2.11 of the textbook, the series is uniformly convergent on $[0, a]$ for any $0 < a < e^2$. □

Alternate proof. If we consider

$$\begin{aligned} f_k(x) &:= \sum_{n=1}^{k} e^{-2n}x^n, \\ f(x) &:= \sum_{n=1}^{\infty} e^{-2n}x^n, \end{aligned}$$

then we claim that $\{f_k\}$ converges uniformly to f on the interval $[0, a]$ for any $0 \le a < e^2$. Let $\epsilon > 0$ be given. First, suppose $x \in (0, a]$ (so that we have $x^0 = 1$; notice that 0^0 is undefined). Then we can write

$$\begin{aligned} f_k(x) &= \sum_{n=1}^{k} e^{-2n}x^n \\ &= \sum_{n=0}^{k} e^{-2n}x^n - e^{-2(0)}x^0 \\ &= \frac{1 - e^{-2(k+1)}x^{k+1}}{1 - e^{-2}x} - 1 \end{aligned}$$

we claim that $\{f_k\}$ converges uniformly to the infinite sum

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} e^{-2n}x^n \\ &= \sum_{n=0}^{\infty} e^{-2n}x^n - e^{-2(0)}x^0 \\ &= \frac{1}{1 - e^{-2x}} - 1 \end{aligned}$$

for any $x \in (0, a]$. Now choose $K > \log_{e^{-2}a}((1 - e^{-2}a)\epsilon) - 1$. If $k \ge K$, then, by the reverse triangle inequality and since

$\log_{e^{-2}a}(x)$ is a decreasing function of x, we obtain

$$\begin{aligned}
|f_k(x) - f(x)| &= \left| \left(\frac{1 - e^{-2(k+1)} x^{k+1}}{1 - e^{-2}x} - 1 \right) - \left(\frac{1}{1 - e^{-2}x} - 1 \right) \right| \\
&= \left| - \frac{e^{-2(k+1)} x^{k+1}}{1 - e^{-2}x} \right| \\
&= \frac{(e^{-2}|x|)^{k+1}}{|1 - e^{-2}x|} \\
&\le \frac{(e^{-2}|x|)^{k+1}}{1 - e^{-2}|x|} \\
&\le \frac{(e^{-2}a)^{k+1}}{1 - e^{-2}a} \\
&\le \frac{(e^{-2}a)^{K+1}}{1 - e^{-2}a} \\
&< \epsilon
\end{aligned}$$

for all $x \in (0, a]$. If $x = 0$, then we have

$$\begin{aligned}
|f_k(0) - f(0)| &= \left| \sum_{n=1}^{k} e^{-2n} x^n - \sum_{n=1}^{\infty} e^{-2n} x^n \right| \\
&= \left| \sum_{n=1}^{k} e^{-2n} 0^n - \sum_{n=1}^{\infty} e^{-2n} 0^n \right| \\
&= |0 - 0| \\
&= 0 \\
&< \epsilon.
\end{aligned}$$

Therefore, f_k converges uniformly to f on $[0, a]$, completing the proof. □

4. Let X be a metric space, $E \subset X$ be closed, and let $\{x_n\}$ be a sequence in X converging to $p \in X$. Suppose $x_n \in E$ for infinitely many $n \in \mathbb{N}$. Show $p \in E$.

   Proof. Suppose by contradiction that we have $p \notin E$. Then we must have $p \in E^c$, where $E^c := X \setminus E$ denotes the set complement of E. Since $E \subset X$ is closed, it follows that $E^c \subset X$ is open. So there exists $\delta > 0$ that satisfies $B(x, \delta) \subset E^c$. Since $\{x_n\}$ converges to p, for all $\epsilon > 0$, there exists $N > 0$ that satisfies $|x_n - p| < \epsilon$ for all $n \ge N$. Choose $\epsilon := \delta$, which means $|x_n - p| < \epsilon = \delta$. This implies $x_n \in B(x, \delta)$ for all $n \ge N$. IN other words, we conclude $x_n \in B(x, \delta) \subset E^c$ for infinitely many $n \in \mathbb{N}$, but this contradicts the assumption of $x_n \in E$ for infinitely many $n \in \mathbb{N}$. So we must concldue $p \in E$. □