MATH 151B discussion, section 002 — Ryan Ta
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Higher-Stakes Homework 3 solutions

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that the first-order partial derivatives of f, f_x and f_y, exist at $(0,0)$. Show, however, that f is not differentiable at $(0,0)$.

Solution. We have

$$\begin{aligned} f_x(0,0) &= \lim_{h_1 \to 0^+} \frac{f(0+h_1,0) - f(0,0)}{h_1} \\ &= \lim_{h_1 \to 0^+} \frac{\frac{h_1^3}{h_1^2+0^2} - 0}{h_1} \\ &= \lim_{h_1 \to 0^+} 1 \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} f_y(0,0) &= \lim_{h_2 \to 0^+} \frac{f(0,0+h_2) - f(0,0)}{h_2} \\ &= \lim_{h_1 \to 0^+} \frac{\frac{0^3}{0^2+h_2^2} - 0}{h_2} \\ &= \lim_{h_1 \to 0^+} 0 \\ &= 0, \end{aligned}$$

meaning that the first-order partial derivatives f_x, f_y exist at $(0,0)$. Next, we will show that f is not differentiable at the origin. Choose for instance the line $h_2 := h_1$, and consider $a := f_x(0,0) = 1$ and $b := f_y(0,0) = 0$, so that the linear map

$$\begin{aligned} A(h_1,h_2) &= ah_1 + bh_2 \\ &= f_x h_1 + f_y h_1 \\ &= (f_x + f_y)h_1 \\ &= (1+0)h_1 \\ &= h_1. \end{aligned}$$

Then, along the path $h_2 := h_1$, we have

$$\begin{aligned} \lim_{(h_1,h_2)\to(0,0)} \frac{|f(h_1,h_2) - f(0,0) - A(h_1,h_2)|}{\|(h_1,h_2)\|} &= \lim_{(h_1,h_2)\to(0,0)} \frac{|\frac{h_1^3}{h_1^2+h_2^2} - 0 - h_1|}{\sqrt{h_1^2+h_2^2}} \\ &= \lim_{(h_1,h_2)\to(0,0)} \frac{|\frac{h_1^3}{h_1^2+h_2^2} - h_1|}{\sqrt{h_1^2+h_2^2}} \\ &= \lim_{(h_1,h_2)\to(0,0)} \frac{|h_1^3 - h_1(h_1^2+h_2^2)|}{(h_1^2+h_2^2)^{\frac{3}{2}}} \\ &= \lim_{(h_1,h_2)\to(0,0)} \frac{|-h_1h_2^2|}{(h_1^2+h_2^2)^{\frac{3}{2}}} \\ &= \lim_{h_1\to 0} \frac{h_1h_1^2}{(h_1^2+h_1^2)^{\frac{3}{2}}} \\ &= \lim_{h_1\to 0} \frac{h_1^3}{(2h_1^2)^{\frac{3}{2}}} \\ &= \lim_{h_1\to 0} \frac{1}{2} \\ &= \frac{1}{2} \\ &\neq 0 \end{aligned}$$

and so we conclude by Definition 8.3.1 of the Lebl textbook that f is not differentiable at $(0,0)$. □

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} x^2+y^2 & \text{if } x \text{ and } y \text{ are rational} \\ 0 & \text{otherwise.} \end{cases}$$

Find all the points $(x,y) \in \mathbb{R}^2$ (if any) where f is differentiable. Justify your answer.

Solution. First, we claim that f is differentiable at the point $(0,0)$. We can choose $a = 0$ and $b = 0$, so that the linear map becomes $A(h_1,h_2) = ah_1 + bh_2 = 0h_1 + 0h_2 = 0$, and so we have

$$\begin{aligned} \lim_{(h_1,h_2)\to(0,0)} \frac{|f(h_1,h_2) - f(0,0) - A(h_1,h_2)|}{\|(h_1,h_2)\|} &= \lim_{(h_1,h_2)\to(0,0)} \frac{|(h_1^2+h_2^2) - 0 - 0|}{\sqrt{h_1^2+h_2^2}} \\ &= \lim_{(h_1,h_2)\to(0,0)} \sqrt{h_1^2+h_2^2} \\ &= \sqrt{0^2+0^2} \\ &= 0. \end{aligned}$$

For the case $(h_1,h_2) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$, we can choose $a = 0$ and $b = 0$, so that the linear map becomes $A(h_1,h_2) = ah_1 + bh_2 = 0h_1 + 0h_2 = 0$, and so we have

$$\begin{aligned} \lim_{(h_1,h_2)\to(0,0)} \frac{|f(h_1,h_2) - f(0,0) - A(h_1,h_2)|}{\|(h_1,h_2)\|} &= \lim_{(h_1,h_2)\to(0,0)} \frac{|0 - 0 - 0|}{\sqrt{h_1^2+h_2^2}} \\ &= \lim_{(h_1,h_2)\to(0,0)} 0 \\ &= 0. \end{aligned}$$

Therefore, f is differentiable at $(0,0)$. Next, we claim that f is not differentiable at any $(x,y) \neq (0,0)$. The contrapositive of Proposition 8.3.5 of the Lebl textbook asserts that it suffices for us to prove that f is not continuous at any $(x,y) \neq (0,0)$. To this end, notice that, for all $(x,y) \neq (0,0)$, we have $f(x,y) \le x^2+y^2$. Consider a sequence $\{(x_n,y_n)\}$ of rational 2-tuples $(x_n,y_n) \in \mathbb{Q}^2$ and a sequence $\{(z_n,w_n)\}$ of irrational 2-tuples $(z_n,w_n) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$, both of which converge to some point $(x_0,y_0) \neq (0,0)$. Then $\{f(x_n,y_n)\}$ converges to $x_n^2 + y_n^2$ and $\{f(z_n,w_n)\}$ converges to 0, and these limits are unequal for any $(x_0,y_0) \neq (0,0)$. Therefore, f is not continuous, and hence not differentiable, at any $(x,y) \neq (0,0)$. □

3. Let $f : (-1,1) \to \mathbb{R}$ be continuously differentiable, so that $f(0) = 0, f'(0) \neq 0$. Show that the function $F(x,y) = (u(x,y), v(x,y))$ given by

$$\begin{aligned} u(x,y) &= f(x) \\ v(x,y) &= -y + xf(x). \end{aligned}$$

is locally invertible near $(0,0)$ and its inverse has the form

$$\begin{aligned} x(u,v) &= g(u) \\ y(u,v) &= -v + ug(u). \end{aligned}$$

Solution. We can rewrite $F : \mathbb{R}^2 \to \mathbb{R}$ as $F(x,y) = (f(x), -y + xf(x))$. The Jacobian matrix is

$$\begin{aligned} F'(x,y) &= \begin{bmatrix} \frac{\partial}{\partial x} f(x) & \frac{\partial}{\partial y} f(x) \\ \frac{\partial}{\partial x}(-y + xf(x)) & \frac{\partial}{\partial y}(-y + xf(x)) \end{bmatrix} \\ &= \begin{bmatrix} f'(x) & 0 \\ f(x) + xf'(x) & -1 \end{bmatrix}. \end{aligned}$$

At the origin $(0,0) \in \mathbb{R}^2$, we have

$$F'(0,0) = \begin{bmatrix} f'(0) & 0 \\ f(0) & -1 \end{bmatrix},$$

whose determinant is

$$\begin{aligned} \det F'(0,0) &= \det \begin{bmatrix} f'(0) & 0 \\ f(0) & -1 \end{bmatrix} \\ &= f'(0)(-1) - 0f(0) \\ &= -f'(0). \end{aligned}$$

Since we assumed $f'(0) \neq 0$, it follows that we have $\det F'(0,0) \neq 0$, meaning that the $F'(0,0)$ is invertible. By the Inverse Function Theorem, there exists open sets $V, W \subset \mathbb{R}^2$ such that $p \in V \subset U$, $f(V) = W$, and $F|_V$ is one-to-one. In other words, F is locally invertible; there exists a function $g : W \to V$ defined by $g(u,v) := (f|_V)^{-1}(u,v)$. That said, we have

$$\begin{aligned} x(u,v) &= x \\ &= f^{-1}(f(x)) \\ &= f^{-1}(u(x,y)) \\ &= g(u(x,y)) \\ &= g(u). \end{aligned}$$

From $v(x,y) = -y + xf(x)$, or abbreviated $v = -y + xf(x)$, we can algebraically rearrange this equation to say $y = -v + xf(x)$. So we have

$$\begin{aligned} y(u,v) &= y \\ &= -v + xf(x) \\ &= -v + g(u)f(g(u)) \\ &= -v + g(u)f(f^{-1}(u)) \\ &= -v + g(u)u \\ &= -v + ug(u), \end{aligned}$$

as desired. □

4. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x,y,z) = x^2 y + 2e^x + z.$$

Prove there exists a differentiable function g defined in some neighborhood $\mathcal{B} \subset \mathbb{R}^2$ of $(1,-2)$ such that $g(1,-2) = 0$ and

$$f(g(y,z), y, z) = 0$$

for all $(y,z) \in \mathcal{B}$. Furthermore, evaluate $\frac{\partial g}{\partial y}(1,-2)$ and $\frac{\partial g}{\partial z}(1,-2)$.

Solution. The Jacobian matrix is

$$\begin{aligned} f'(x,y,z) &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial}{\partial x}(x^2y + 2e^x + z) & \frac{\partial}{\partial y}(x^2y + 2e^x + z) & \frac{\partial}{\partial z}(x^2y + 2e^x + z) \end{bmatrix} \\ &= \begin{bmatrix} 2xy + 2e^x & x^2 & 1 \end{bmatrix}. \end{aligned}$$

At the point $(0,1,-2) \in \mathbb{R}^3$, we have

$$f'(0,1,-2) = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix},$$

which is a nonzero matrix. By the Implicit Function Theorem, there exist an open set $\mathcal{B} \subset \mathbb{R}^2$ with $(1,-2) \in \mathbb{R}^2$, an open set $\mathcal{B}' \subset \mathbb{R}$ with $0 \in \mathcal{B}'$, with $\mathcal{B} \times \mathcal{B}' \subset U$, and a $C^1(W)$ map $g : \mathcal{B} \to \mathcal{B}'$, with $g(1,2) = 0$, and, for all $(y,z) \in \mathcal{B}$, the point $g(y,z)$ is the unique point in $\mathcal{B}$ such that $f(g(y,z), y, z) = 0$.

Next, we will evaluate $\frac{\partial g}{\partial y}(1,-2)$ and $\frac{\partial g}{\partial z}(1,-2)$. Since the point $g(y,z)$ is unique, we can write $x = g(y,z)$, allowing us to write

$$\begin{aligned} 0 &= f(g(y), y, z) \\ &= g(y,z)^2 y + 2e^{g(y,z)} + z. \end{aligned}$$

While we can apply the remaining conclusions of the Implicit Function Theorem to compute the partial derivatives, I find it easier if we employ implicit differentiation. That said, we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial y} 0 \\ &= \frac{\partial}{\partial y}(g(y,z)^2 y + 2e^{g(y,z)} + z) \\ &= 2g(y,z)\frac{\partial g}{\partial y}(y,z)y + 2e^{g(y,z)}\frac{\partial g}{\partial y}(y,z) + 0 \\ &= 2(yg(y,z) + e^{g(y,z)})\frac{\partial g}{\partial y}(y,z) \end{aligned}$$

and

$$
\begin{aligned}
0 &= \frac{\partial}{\partial z} 0 \\
&= \frac{\partial}{\partial z}(g(y,z)^2 y + 2e^{g(y,z)} + z) \\
&= 2g(y,z)\frac{\partial g}{\partial z}(y,z)y + 2e^{g(y,z)}\frac{\partial g}{\partial z}(y,z) + 1 \\
&= 2(yg(y,z) + e^{g(y,z)})\frac{\partial g}{\partial z}(y,z) + 1.
\end{aligned}
$$

At the point $(1,-2)$, we have

$$
\begin{aligned}
0 &= 2((-2)g(1,-2) + e^{g(1,-2)})\frac{\partial g}{\partial y}(1,-2) \\
&= 2((-2)(0) + e^{0})\frac{\partial g}{\partial y}(1,-2) \\
&= 2\frac{\partial g}{\partial y}(1,-2)
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= 2((-2)g(1,-2) + e^{g(1,-2)})\frac{\partial g}{\partial z}(1,-2) + 1 \\
&= 2((-2)(0) + e^{0})\frac{\partial g}{\partial y}(1,-2) + 1 \\
&= 2\frac{\partial g}{\partial y}(1,-2) + 1,
\end{aligned}
$$

from which we obtain $\frac{\partial g}{\partial y}(1,-2) = \boxed{0}$ and $\frac{\partial g}{\partial y}(1,-2) = \boxed{-\frac{1}{2}}$, respectively. □