MATH 151B discussion, section 002
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Homework 1 solutions

1. (Exercise 6.1.2)

a) Find the pointwise limit $\frac{e^{\frac{x}{n}}}{n}$ for $x \in \mathbb{R}$.

Answer. First, we compute the limit

$$\begin{aligned}\lim_{n\to\infty} \frac{e^{\frac{x}{n}}}{n} &= \left(\lim_{n\to\infty} \frac{1}{n}\right)\left(\lim_{n\to\infty} \frac{e^{\frac{x}{n}}}{n}\right) \\ &= 0 \cdot 1 \\ &= 0.\end{aligned}$$

for all $x \in \mathbb{R}$. Next, we will prove that $\frac{e^{\frac{x}{n}}}{n}$ converges pointwise to the limit we found. Let $\epsilon > 0$ be given. Choose $N > \ln(\epsilon)$. If $n \geq N$, then we have

$$\begin{aligned}\left|\frac{e^{\frac{x}{n}}}{n} - 0\right| &= \frac{e^{\frac{x}{n}}}{n} \\ &\leq \frac{e^x}{n} \\ &\leq \frac{e^x}{N} \\ &= \epsilon.\end{aligned}$$

So $\frac{e^{\frac{x}{n}}}{n}$ converges pointwise to 0 for all $x \in \mathbb{R}$. □

b) Is the limit uniform on $\mathbb{R}$?

Answer. No. Choose $\epsilon := \frac{1}{2}$. Consider the sequence $\{x_n\}$ given by $x_n := n^2$. Next, we will prove the inequality $n^2 < e^n$ for all positive integers n, which is an example of showing that exponential functions grow faster than polynomials. I will choose to prove by induction, although there are also many other ways of showing this inequality. The base case holds true because we have $1^2 = 1 < e = e^1$. For the induction step, we assume $k^2 < e^k$ and obtain

$$\begin{aligned}(k+1)^2 &= k^2 + 2k + 1 \\ &< e^k + 2k + 1 \\ &< e^k(1 + e^{-k}(2k+1)) \\ &\leq e^k e \\ &= e^{k+1},\end{aligned}$$

which would complete the induction proof provided that we show our claim $e^{-k}(2k+1) \leq e - 1$. Indeed, consider the smooth function $e^{-x}(2x+1)$. Its first derivative is $e^{-x}(-2x+1)$, which is negative for all $x > \frac{1}{2}$. So the function $e^{-x}(2x+1)$ is strictly decreasing for all $x > \frac{1}{2}$. Furthermore, the function $e^{-x}(2x+1)$ attains a maximum of $\frac{2}{\sqrt{e}}$ at $x = \frac{1}{2}$, and so at any integer $k > 1$ we obtain

$$\begin{aligned}e^{-k}(2k+1) &< \frac{2}{\sqrt{e}} \\ &< e - 1,\end{aligned}$$

thereby establishing our claim. (As an alternative proof of $n^2 < e^n$ without resorting to first-year calculus, consider showing by induction the inequality $n^2 \leq 2^n$ for all positive integers $n \neq 3$, and combine this with $2^n < e^n$, which is true because we have $2 < e$ and x^n is an increasing function of x.) Therefore, $n^2 < e^n$ holds for all positive integers n, and so we have

$$\begin{aligned}\left|\frac{e^{\frac{xn}{n}}}{n}-0\right| &= \frac{e^{\frac{xn}{n}}}{n}\\ &= \frac{e^{\frac{n^2}{n}}}{n}\\ &= \frac{e^n}{n}\\ &> \frac{n^2}{n}\\ &= n\\ &\geq 1\\ &> \frac{1}{2}\\ &= \epsilon.\end{aligned}$$

So the limit is not uniform on $\mathbb{R}$. □

c) Is the limit uniform on $[0, 1]$?

Answer. Let $\epsilon > 0$ be given. Choose $N > \frac{e}{\epsilon}$. Note that we have $\frac{x}{n} \leq \frac{1}{n}$ for all positive integers n and that e^x is an increasing function of x, and so we have $e^{\frac{x}{n}} \leq e^{\frac{1}{n}}$. Therefore, if $n \geq N$, then we have

$$\begin{aligned}\left|\frac{e^{\frac{x}{n}}}{n}-0\right| &= \frac{e^{\frac{x}{n}}}{n}\\ &\leq \frac{e^{\frac{1}{n}}}{n}\\ &\leq \frac{e}{n}\\ &\leq \frac{e}{N}\\ &< \epsilon\end{aligned}$$

for all $0 \leq x \leq 1$. Since N depends only on ϵ (and not on x), we conclude that the limit is uniform on $[0, 1]$. □

2. (Exercise 6.1.6) Find an example of a sequence of functions $\{f_n\}$ and $\{g_n\}$ that converge uniformly to some f and g on some set A, but such that $\{f_n g_n\}$ (the multiple) does not converge uniformly to fg on A.

Hint: Let $A := \mathbb{R}$, let $f(x) := g(x) := x$. You can even pick $f_n = g_n$.

Answer. Let $A := \mathbb{R}$. Choose $f_n(x) := g_n(x) := x + \frac{1}{n}$. Then $\{f_n\}$ converges uniformly to $f(x) := x$, and $\{g_n\}$ converges uniformly to $g(x) := x$. (Perhaps you can prove these statements yourself as quick exercises.) We have

$$\begin{aligned}|(f_n g_n)(x) - (fg)(x)| &= |f_n(x)g_n(x) - f(x)g(x)|\\ &= \left|\left(x+\frac{1}{n}\right)\left(x+\frac{1}{n}\right) - xx\right|\\ &= \left|\left(x+\frac{1}{n}\right)^2 - x^2\right|\\ &= \left|\frac{2}{n}x + \frac{1}{n^2}\right|\\ &= \frac{|2nx+1|}{n^2}\\ &\geq \frac{2nx+1}{n^2}\end{aligned}$$

for all $x \in \mathbb{R}$. Consider a sequence $\{x_n\}$ given by $x_n := n$. Choose $\epsilon := 2$. Then we have

$$\begin{aligned}|(f_n g_n)(x_n) - (fg)(x_n)| &\geq \frac{2nx_n+1}{n^2}\\ &= \frac{2n(n)+1}{n^2}\\ &= 2 + \frac{1}{n^2}\\ &> 2\\ &= \epsilon.\end{aligned}$$

So $\{f_n g_n\}$ does not coinverge uniformly to fg on $A := \mathbb{R}$. □

3. (Exercise 6.1.7) Suppose there exists a sequence of functions $\{g_n\}$ uniformly converging to 0 on A. Now suppose we have a sequence of functions $\{f_n\}$ and a function f on A such that

$$|f_n(x) - f(x)| \leq g_n(x)$$

for all $x \in A$. Show that $\{f_n\}$ converges uniformly to f on A.

Proof. Since $\{g_n\}$ converges uniformly to 0 on A, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$|g_n(x) - 0| < \epsilon$$

for all $x \in A$. Note that the inequality

$$|f_n(x) - f(x)| \leq g_n(x)$$

necessarily implies that $g_n(x)$ is nonnegative; in other words, we have $|g_n(x)| = g_n(x)$, and so we can say in fact

$$\begin{aligned} g_n(x) &= |g_n(x)| \\ &= |g_n(x) - 0| \\ &< \epsilon. \end{aligned}$$

Finally, we can combine our previous inequalities to conclude

$$\begin{aligned} |f_n(x) - f(x)| &\leq g_n(x) \\ &< \epsilon \end{aligned}$$

for all $x \in A$. Therefore, $\{f_n\}$ converges uniformly to f on A. □

4. (Exercise 6.1.9) Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of increasing functions (that is, $f_n(x) \geq f_n(y)$ whenever $x \geq y$). Suppose $f_n(0) = 0$ and $\lim_{n\to\infty} f_n(1) = 0$. Show that $\{f_n\}$ converges uniformly to 0.

Proof. Since we have $\lim_{n\to\infty} f_n(1) = 0$, for all $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that for all $n \geq N$ we have

$$|f_n(1) - 0| < \epsilon.$$

Since f_n is increasing, we have $f_n(x) \geq f_n(0)$ for all $0 \leq x \leq 1$, which means $f_n(x) - f_n(0)$ is nonnegative, and so we can write

$$|f_n(x) - f_n(0)| = f_n(x) - f_n(0).$$

Since we assume $f_n(0) = 0$, we have in fact

$$\begin{aligned} |f_n(x) - 0| &= |f_n(x) - f_n(0)| \\ &= f_n(x) - f_n(0) \\ &= f_n(x) - 0 \\ &= f_n(x) \end{aligned}$$

for all $0 \leq x \leq 1$. At $x = 1$, we have

$$\begin{aligned} f_n(1) &= |f_n(1) - 0| \\ &< \epsilon. \end{aligned}$$

Since f_n is increasing, we have

$$f_n(x) \leq f_n(1)$$

for all $0 \leq x \leq 1$. So for all $n \geq N$ we conclude

$$\begin{aligned} |f_n(x) - 0| &= f_n(x) \\ &\leq f_n(1) \\ &< \epsilon \end{aligned}$$

for all $0 \leq x \leq 1$. Therefore, $\{f_n\}$ converges uniformly to 0. □

5. (Exercise 6.1.10) Let $\{f_n\}$ be a sequence of functions defined on $[0, 1]$. Suppose there exists a sequence of distinct numbers $x_n \in [0, 1]$ such that

$$f_n(x_n) = 1.$$

Prove or disprove the following statements:

a) True or false: There exists $\{f_n\}$ as above that converges to 0 pointwise.

Answer. True. Consider, for example, the sequence $\{x_n\}$ given by $x_n := \frac{1}{n}$ and the sequence of functions $\{f_n\}$ given by

$$f_n(x) = \begin{cases} nx & \text{if } 0 \le x < \frac{1}{2n^2}, \\ \frac{1}{nx} & \text{if } \frac{1}{2n^2} \le x \le 1, \end{cases}$$

Then we would have $f_n(x_n) = 1$ for any $x_n \in [0, 1]$. Also, let $\epsilon > 0$ and choose $N > \max\{\frac{1}{2\epsilon}, \frac{1}{\epsilon x}\}$ (that is, choose N that satisfies $N > \frac{1}{2\epsilon}$ and $N > \frac{1}{\epsilon x}$). If $n \ge N$ and $0 \le x < \frac{1}{2n^2}$, then we would have

$$\begin{aligned} |f_n(x) - f(x)| &= |nx - 0| \\ &= nx \\ &< n\frac{1}{2n^2} \\ &= \frac{1}{2n} \\ &\le \frac{1}{2N} \\ &< \epsilon \end{aligned}$$

for all $0 \le x \le 1$. If $n \ge N$ and $\frac{1}{2n^2} \le x \le 1$, then we would have

$$\begin{aligned} |f_n(x) - f(x)| &= \left|\frac{1}{nx} - 0\right| \\ &= \frac{1}{nx} \\ &\le \frac{1}{Nx} \\ &< \epsilon \end{aligned}$$

for all $0 \le x \le 1$. So $\{f_n\}$ converges pointwise to 0 for all $0 \le x \le 1$. By the way, my solution is only one example; please construct your own example of f_n that satisfies all the given conditions! □

b) True or false: There exists $\{f_n\}$ as above that converges to 0 uniformly on $[0, 1]$.

Answer. False. Suppose to the contrary that there exists $\{f_n\}$ as above that converges uniformly to 0. Then there exists $N \in \mathbb{N}$ such that, if $n \ge N$, then we would have $|f_n(x) - 0| < \epsilon$ for all $0 \le x \le 1$. Choose for instance $\epsilon := \frac{1}{2}$. Since each x_n is in $[0, 1]$ and we assumed $|f_n(x_n)| = 1$, we have in fact

$$\begin{aligned} 1 &= |f_n(x_n)| \\ &= |f_n(x_n) - 0| \\ &< \epsilon \\ &= \frac{1}{2}, \end{aligned}$$

which is a contradiction. □