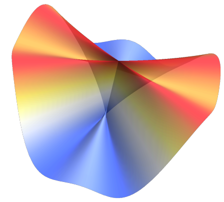
UC RIVERSIDE
Department of Mathematics

MATH 151B - Advanced Calculus



Homework 2

Due date: Tuesday, January 19 at 11:59pm

Notice that due to the MLK holiday the assignment is due on Tuesday, not Monday

Instructions:

- You are encourage to work in group but you have to write your individual solution
- You can use any book, article or web-based mathematical material or computational software
- You can ask Chulan, Ryan, or Estela questions
- Chegg, Math Stack Exchange, or any other source where you can copy solutions is not allowed
- The homework needs to be typeset in "LaTeX" .

1. (Exercise 6.2.1) While uniform convergence preserves continuity, it does not preserve differentiability. Find an explicit example of a sequence of differentiable functions on $[-1,1]$ that converge uniformly to a function f such that f is not differentiable.

   Hint: There are many possibilities, simplest is perhaps to is to consider $\sqrt{x^2+(\frac{1}{n})^2}$*. Another one is to* <u>*combine*</u> *in an appropriate way the functionns* $|x|$ *and* $\frac{n}{2}x^2+\frac{1}{2n}$ *to construct the sequence . Show that these functions are differentiable, converge uniformly, and then show that the limit is not differentiable.*

2. (Exercise 6.2.3) Let $f\colon [0,1] \to \mathbb{R}$ be a Riemann integrable (hence bounded) function. Find

$$\lim_{n\to\infty} \int_0^1 \frac{f(x)}{n}\, dx.$$

3. (Exercise 6.2.5) Find an example of a sequence of continuous functions on $(0, 1)$ that converges pointwise to a continuous function on $(0, 1)$, but the convergence is not uniform.

4. (Exercise 6.2.6) True/False; prove or find a counterexample to the following statement:

   If $\{f_n\}$ is a sequence of everywhere discontinuous functions on $[0,1]$ that converge uniformly to a function f, then f is everywhere discontinuous.

Suppose $f\colon [0,1] \to \mathbb{R}$ is Riemann integrable. For the following two exercises define the number

$$\|f\|_{L^1} := \int_0^1 |f(x)|\, dx.$$

It is true that $|f|$ is integrable whenever f is, see Exercise 5.2.15. The number is called the L^1-norm and defines another very common type of convergence called the L^1-convergence.

5. (Exercise 6.2.8) Suppose $\{f_n\}$ is a sequence of Riemann integrable functions on $[0,1]$ that converges uniformly to 0. Show that

$$\lim_{n\to\infty} \|f_n\|_{L^1} = 0.$$

6. (Exercise 6.2.9) Find a sequence $\{f_n\}$ of Riemann integrable functions on $[0,1]$ converging pointwise to 0, but
$$\lim_{n\to\infty} \|f_n\|_{L^1} \text{ does not exist (is } \infty\text{).}$$

7. (Exercise 6.2.15. a))

   We say that a sequence of functions $f_n \colon \mathbb{R} \to \mathbb{R}$ converges uniformly on compact subsets

   if for every $k \in \mathbb{N}$, the sequence $\{f_n\}$ converges uniformly on $[-k,k]$.

   a) Prove that if $f_n \colon \mathbb{R} \to \mathbb{R}$ is a sequence of continuous functions converging uniformly on compact subsets, then the limit is continuous.

8. (Exercise 6.2.15. b)) (continuation of the previous exercise)

   b) Prove that if $f_n\colon \mathbb{R} \to \mathbb{R}$ is a sequence of functions Riemann integrable on any closed and bounded interval $[a,b]$, and converging uniformly on compact subsets to an $f\colon \mathbb{R} \to \mathbb{R}$, then for any interval $[a,b]$, we have f is Riemann integrable on $[a,b]$, and

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

9. (Exercise 6.2.18. a))

   a) Find a sequence of Lipschitz continuous functions on $[0, 1]$ whose uniform limit is $\sqrt{x}$, which is a non-Lipschitz function. See **Definition 3.4.7.** in the textbook for the definition of a "Lipschitz" function.

10. (Exercise 6.2.18. b)) (continuation of the previous exercise)

    b) On the other hand, show that if $f_n \colon S \to \mathbb{R}$ are Lipschitz with a uniform constant K (meaning all of them satisfy the definition with the same constant) and $\{f_n\}$ converges pointwise to $f \colon S \to \mathbb{R}$, then the limit f is a Lipschitz continuous function with Lipschitz constant K.