Homework 2 solutions

1. (Exercise 6.2.1) While uniform convergence preserves continuity, it does not preserve differentiability. Find an explicit example of a sequence of differentiable functions on $[-1, 1]$ that converge uniformly to a function f such that f is not differentiable.

   Hint: There are many possibilities, simplest is perhaps to combine $|x|$ and $\frac{n}{2}x^2 + \frac{1}{2n}$, another is to consider $\sqrt{x^2 + \frac{1}{n^2}}$. Show that these functions are differentiable, converge uniformly, and then show that the limit is not differentiable.

   Answer. Define for example $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$, which is differentiable on $[-1, 1]$ and $f(x) = |x|$, which is not differentiable on $[-1, 1]$. First, we claim $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for all $a, b \in \mathbb{R}$. The true statement $\sqrt{ab} \ge 0$ implies

$$\begin{aligned}(\sqrt{a} + \sqrt{b})^2 &= (\sqrt{a})^2 + 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 \\ &= a + 2\sqrt{ab} + b \\ &\ge a + 2 \cdot 0 + b \\ &= a + b \\ &= (\sqrt{a+b})^2.\end{aligned}$$

   Since $\sqrt{x}$ is an increasing function of x for all $x \ge 0$, we can apply the square root to both sides of our previous inequality in order to obtain the desired claim. Also, since we have $\frac{1}{n^2} \ge 0$ and $\sqrt{x}$ is an increasing function of x for all $x \ge 0$, we have

$$\begin{aligned}\sqrt{x^2 + \frac{1}{n^2}} - |x| &\ge \sqrt{x^2} - |x| \\ &= |x| - |x| \\ &= 0,\end{aligned}$$

   which means $\left|\sqrt{x^2 + \frac{1}{n^2}} - |x|\right| = \sqrt{x^2 + \frac{1}{n^2}} - |x|$. Applying our results, we obtain

$$\begin{aligned}|f_n(x) - f(x)| &= \left|\sqrt{x^2 + \frac{1}{n^2}} - |x|\right| \\ &= \sqrt{x^2 + \frac{1}{n^2}} - |x| \\ &\le \sqrt{x^2} + \sqrt{\frac{1}{n^2}} - |x| \\ &= |x| + \frac{1}{n} - |x| \\ &= \frac{1}{n}\end{aligned}$$

   for all $-1 \le x \le 1$. Finally, let $\epsilon > 0$. Choose $N > \frac{1}{\epsilon}$. If $n \ge N$, then we would have

$$\begin{aligned}|f_n(x) - f(x)| &\le \frac{1}{n} \\ &\le \frac{1}{N} \\ &< \epsilon\end{aligned}$$

   for all $-1 \le x \le 1$. So $\{f_n\}$ converges uniformly to f on $[-1, 1]$. □

2. (Exercise 6.2.3) Let $f\colon [0, 1] \to \mathbb{R}$ be a Riemann integrable (hence bounded) function. Find $\lim_{n\to\infty} \int_0^1 \frac{f(x)}{n}\, dx$.

   Answer. First, we will compute the limit (optional step). Since f is a bounded function on $[0, 1]$, there exists $M > 0$ such that f satisfies $|f(x)| \le M$ for all $0 \le x \le 1$. We have

$$-M \le f(x) \le M$$

   for all $x \in [0, 1]$. We can integrate over $[0, 1]$ to obtain

$$\int_0^1 -M\, dx \le \int_0^1 f(x)\, dx \le \int_0^1 M\, dx,$$

which is

$$-M \leq \int_0^1 f(x)\,dx \leq M.$$

Multiply by $\frac{1}{n}$ to conclude

$$-\frac{M}{n} \leq \int_0^1 \frac{f(x)}{n}\,dx \leq \frac{M}{n}.$$

Now send $n \longrightarrow \infty$ and invoke the Squeeze Theorem to conclude

$$\lim_{n\to\infty} \int_0^1 \frac{f(x)}{n}\,dx = 0,$$

as desired.

Next, we will prove the limit (mandatory step). Let $\epsilon > 0$ be given, and choose $N > \frac{M}{\epsilon}$. Since f is a bounded function on $[0, 1]$, there exists $M > 0$ such that f satisfies $|f(x)| \leq M$ for all $0 \leq x \leq 1$. If $n \geq N$, we have

$$\begin{aligned}\left|\int_0^1 \frac{f(x)}{n}\,dx - 0\right| &\leq \int_0^1 \frac{|f(x)|}{n}\,dx \\ &\leq \int_0^1 \frac{M}{n}\,dx \\ &= \frac{M}{n} \\ &\leq \frac{M}{N} \\ &< \epsilon,\end{aligned}$$

as desired. □

3. (Exercise 6.2.5) Find an example of a sequence of continuous functions on $(0, 1)$ that converges pointwise to a continuous function on $(0, 1)$, but the convergence is not uniform.

Answer. Define $\{f_n\}$ by $f_n(x) = \frac{1}{nx}$ and $f(x) = 0$, which are continuous on $(0, 1)$. Let $\epsilon > 0$. Choose $N > \frac{1}{x\epsilon}$. If $n \geq N$, then we have

$$\begin{aligned}|f_n(x) - f(x)| &= \left|\frac{1}{nx} - 0\right| \\ &= \frac{1}{nx} \\ &\leq \frac{1}{Nx} \\ &< \epsilon\end{aligned}$$

for all $0 < x < 1$. So $\{f_n\}$ converges pointwise to 0. However, if we choose $\epsilon := \frac{1}{2}$, and consider $x_n := \frac{1}{n}$. Then we would have

$$\begin{aligned}|f_n(x_n) - f(x_n)| &= \frac{1}{nx_n} \\ &= \frac{1}{n(\frac{1}{n})} \\ &= 1 \\ &> \frac{1}{2} \\ &= \epsilon\end{aligned}$$

for all $0 < x < 1$. So $\{f_n\}$ does not converge uniformly to 0 on $(0, 1)$. □

4. (Exercise 6.2.6) True/False: prove or find a counterexample to the following statement:

   If $\{f_n\}$ is a sequence of everywhere discontinuous functions on $[0, 1]$ that converge uniformly to a function f, then f is everywhere discontinuous.

Answer. False. Define $\{f_n\}$ by

$$f_n(x) := \begin{cases} \frac{1}{n} & \text{if } 0 \leq x \leq 1 \text{ is rational,} \\ 0 & \text{if } 0 \leq x \leq 1 \text{ is irrational,} \end{cases}$$

which is everywhere discontinuous on $[0,1]$. Let $\epsilon > 0$ be given, and choose $N > \frac{1}{\epsilon}$. If $n \geq N$, then we would have

$$\begin{aligned} |f_n(x) - 0| &= f_n(x) \\ &\leq \frac{1}{n} \\ &\leq \frac{1}{N} \\ &< \epsilon \end{aligned}$$

for all $0 \leq x \leq 1$. So $\{f_n\}$ converges uniformly to 0. But 0 is a continuous function on $[0,1]$. □

5. Suppose $f\colon [0,1] \to \mathbb{R}$ is Riemann integrable. For the following two exercises define the number

$$\|f\|_{L^1} := \int_0^1 |f(x)|\, dx.$$

It is true that $|f|$ is integrable whenever f is, see Exercise 5.2.15. The number is called the L^1-norm and defines another very common type of convergence called the L^1-convergence.

(a) (Exercise 6.2.8) Suppose $\{f_n\}$ is a sequence of Riemann integrable functions on $[0,1]$ that converges uniformly to 0. Show that

$$\lim_{n\to\infty} \|f_n\|_{L^1} = 0.$$

Answer. Let $\epsilon > 0$ be given. Since $\{f_n\}$ converges uniformly to 0, there exists $N \in \mathbb{N}$ such that, if $n \geq N$, then we have $|f_n(x) - 0| < \epsilon$ for all $0 \leq x \leq 1$. So we have

$$\begin{aligned} \big|\|f_n\|_{L^1} - 0\big| &= \|f_n\|_{L^1} \\ &= \int_0^1 |f_n(x)|\, dx \\ &= \int_0^1 |f_n(x) - 0|\, dx \\ &< \int_0^1 \epsilon\, dx \\ &= \epsilon, \end{aligned}$$

which proves $\lim\limits_{n\to\infty} \|f_n\|_{L^1} = 0$.

Alternatively, you can apply Theorem 6.2.4 of the Lebl textbook to write a shorter proof. □

6. (b) (Exercise 6.2.9) Find a sequence $\{f_n\}$ of Riemann integrable functions on $[0,1]$ converging pointwise to 0, but

$$\lim_{n\to\infty} \|f_n\|_{L^1} \text{ does not exist (is } \infty).$$

Answer. Choose for instance (Estela's example)

$$f_n(x) := \begin{cases} 2n^3 x & \text{if } 0 \leq x < \frac{1}{2n}, \\ -2n^3(x - \frac{1}{n}) & \text{if } \frac{1}{2n} \leq x < \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} \leq x \leq 1. \end{cases}$$

The graph of f_n on $[0,1]$ is a triangle with vertices $(0,0), (\frac{1}{2n}, n^2), (\frac{1}{n}, 0)$ for $0 \leq x < \frac{1}{n}$ and the zero function for $\frac{1}{n} \leq x \leq 1$. And this example of $\{f_n\}$ converges pointwise to 0. The L^1-norm of f_n is

$$\begin{aligned} \|f_n\|_{L^1} &= \int_0^1 |f_n(x)|\, dx \\ &= \frac{1}{2} \cdot \frac{1}{n} \cdot n^2 \\ &= \frac{n}{2} \\ &< \infty, \end{aligned}$$

meaning that each f_n is Riemann integrable on $[0,1]$. This also implies

$$\begin{aligned} \lim_{n\to\infty} \|f\| &= \lim_{n\to\infty} \frac{n}{2} \\ &= \infty, \end{aligned}$$

meaning that f is not Riemann integrable. □

7. We say that a sequence of functions $f_n\colon \mathbb{R} \to \mathbb{R}$ converges uniformly on compact subsets if for every $k \in \mathbb{N}$, the sequence $\{f_n\}$ converges uniformly on $[-k, k]$.

   (a) (Exercise 6.2.15. (a)) Prove that if $f_n\colon \mathbb{R} \to \mathbb{R}$ is a sequence of continuous functions converging uniformly on compact subsets, then the limit is continuous.

   Answer. For any $x \in \mathbb{R}$, there exists $k > 0$ such that $x \in [-k, k]$. Since we were given that f is continuous on $[-k, k]$, Theorem 6.2.2 of the Lebl textbook allows us to conclude that f is continuous on $\mathbb{R}$. □

8. (b) Prove that if $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of functions Riemann integrable on any closed and bounded interval $[a, b]$, and converging uniformly on compact subsets to an $f : \mathbb{R} \to \mathbb{R}$, then for any interval $[a, b]$, we have f is Riemann integrable on $[a, b]$, and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx,$$

   as desired.

   Answer. For any $a, b \in \mathbb{R}$, there exists $k > 0$ such that $[a, b] \subset [-k, k]$. Since we were given that $\{f_n\}$ converges uniformly to f on $[a, b]$, Theorem 6.2.4 of the Lebl textbook allows us to conclude that f is Riemann integrable on $\mathbb{R}$ and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx,$$

   as desired. □

9. (a) (Exercise 6.2.18. (a)) Find a sequence of Lipschitz continuous functions on $[0, 1]$ whose uniform limit is $\sqrt{x}$, which is a non-Lipschitz function. See **Definition 3.4.7.** in the textbook for the definition of a "Lipschitz" function.

   Answer. Let $f_n(x) := \sqrt{x + \frac{1}{n}}$ and $f(x) := \sqrt{x}$ for all $0 \le x \le 1$. Then $\{f_n\}$ converges uniformly to f. We have

$$\begin{aligned}
|f_n(x) - f_n(y)| &= \left|\sqrt{x + \frac{1}{n}} - \sqrt{y + \frac{1}{n}}\right| \\
&= \left|\sqrt{x + \frac{1}{n}} - \sqrt{y + \frac{1}{n}}\right| \frac{\left|\sqrt{x + \frac{1}{n}} + \sqrt{y + \frac{1}{n}}\right|}{\left|\sqrt{x + \frac{1}{n}} + \sqrt{y + \frac{1}{n}}\right|} \\
&= \frac{|(x + \frac{1}{n}) - (y + \frac{1}{n})|}{\left|\sqrt{x + \frac{1}{n}} + \sqrt{y + \frac{1}{n}}\right|} \\
&= \frac{|x - y|}{\sqrt{x + \frac{1}{n}} + \sqrt{y + \frac{1}{n}}} \\
&\le \frac{|x - y|}{\sqrt{0 + \frac{1}{n}} + \sqrt{0 + \frac{1}{n}}} \\
&= \frac{\sqrt{n}}{2}|x - y| \\
&= K_n|x - y|
\end{aligned}$$

   for all $0 \le x \le 1$ and $0 \le y \le 1$, meaning that $f_n(x)$ is Lipschitz continuous with $K_n := \frac{\sqrt{n}}{2}$.
   Now we will prove that $f(x)$ is not Lipschitz continuous. Assume to the contrary that $f(x)$ is Lipschitz continuous. Then there exists $K > 0$ such that we have

$$|f(x) - f(y)| \le K|x - y|$$

   for all $0 \le x \le 1$ and $0 \le y \le 1$. By setting $x := 0$ and $y := \frac{1}{n}$, we obtain

$$\begin{aligned}
\left|f(0) - f\left(\frac{1}{n}\right)\right| &= \left|\sqrt{0} - \sqrt{\frac{1}{n}}\right| \\
&= \frac{1}{\sqrt{n}}
\end{aligned}$$

and

$$\left|f(0) - f\left(\frac{1}{n}\right)\right| \leq K\left|0 - \frac{1}{n}\right|$$
$$= \frac{K}{n}.$$

We combine our results to conclude

$$\frac{1}{\sqrt{n}} \leq \frac{K}{n},$$

or equivalently

$$K \geq \sqrt{n},$$

implying that K depends on n (because it is bounded below by $\sqrt{n}$). But this contradicts our assumption that K does not depend on n (because it is a uniform Lipschitz constant). □

(b) (Exercise 6.2.18. (b)) On the other hand, show that if $f_n \colon S \to \mathbb{R}$ are Lipschitz with a uniform constant K (meaning all of them satisfy the definition with the same constant) and $\{f_n\}$ converges pointwise to $f \colon S \to \mathbb{R}$, then the limit f is a Lipschitz continuous function with Lipschitz constant K.

Answer. Let $\epsilon > 0$ be given. Since $\{f_n\}$ converges pointwise to f, there exists $N = N(x, \epsilon) \in \mathbb{N}$ such that, if $n \geq N$, then we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}$$

for all $x \in S$. Also, since each f_n is Lipschitz continuous with a uniform constant K, we have

$$|f_n(x) - f_n(y)| \leq K|x - y|$$

for all $x, y \in S$. So we have

$$\begin{aligned}|f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + K|x - y| + |f_n(y) - f(y)| \\ &< \frac{\epsilon}{2} + K|x - y| + \frac{\epsilon}{2} \\ &= K|x - y| + \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude

$$|f(x) - f(y)| \leq K|x - y|,$$

meaning that f is also Lipschitz continuous. □