Homework 3 solutions

1. Find a closed form of the series

$$\sum_{k=2}^{\infty} kx^{k-2}$$

and the largest set on which this formula is valid.

Solution. We start with the infinite geometric sum

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

for all $-1 < x < 1$. Then we obtain

$$\begin{aligned}\sum_{k=2}^{\infty} kx^{k-1} &= \frac{d}{dx}\left(\sum_{k=2}^{\infty} x^k\right)\\ &= \frac{d}{dx}\left(\sum_{k=0}^{\infty} x^k - 1 - x\right)\\ &= \frac{d}{dx}\left(\frac{1}{1-x} - 1 - x\right)\\ &= \frac{1}{(1-x)^2} - 1.\end{aligned}$$

Furthermore, now assuming $x \neq 0$, we obtain

$$\begin{aligned}\sum_{k=2}^{\infty} kx^{k-2} &= \frac{1}{x}\sum_{k=2}^{\infty} kx^{k-1}\\ &= \frac{1}{x}\left(\frac{1}{(1-x)^2} - 1\right)\\ &= \frac{1}{x(1-x)^2} - \frac{1}{x}\\ &= \frac{2-x}{(1-x)^2}\end{aligned}$$

for all $x \in (-1, 0) \cup (0, 1)$. □

2. (Exercise 6.2.21): Let $f_n(x) := \dfrac{x}{1+(nx)^2}$. Notice that f_n are differentiable functions.

   (a) Show that $\{f_n\}$ converges uniformly to 0.

   Solution. First, we will find the maximum of f_n. We have the first derivative

$$\begin{aligned}f_n'(x) &= \frac{d}{dx}\left(\frac{x}{1+(nx)^2}\right)\\ &= \frac{1-(nx)^2}{(1+(nx)^2)^2}.\end{aligned}$$

   We set $f_n'(x) = 0$ in order to obtain the critical points $x = \pm\frac{1}{n}$, which yield the minimum $f(-\frac{1}{n}) = -\frac{1}{2n}$ and the maximum $f(\frac{1}{n}) = \frac{1}{2n}$. In other words, we obtain

$$-\frac{1}{2n} \leq f_n(x) \leq \frac{1}{2n},$$

   or equivalently

$$|f_n(x)| \leq \frac{1}{2n}.$$

   Now we will apply this inequality. Let $\epsilon > 0$ be given, and choose $N > \frac{1}{2\epsilon}$. If $n \geq N$, then we have

$$\begin{aligned}|f_n(x) - 0| &= |f_n(x)|\\ &\leq \frac{1}{2n}\\ &\leq \frac{1}{2N}\\ &< \epsilon.\end{aligned}$$

   Therefore, $\{f_n\}$ converges uniformly to 0. □

Alternate solution. For all positive integers n, we have

$$\begin{aligned} 0 &\le (nx-1)^2 \\ &= (nx)^2 - 2nx + 1, \end{aligned}$$

which is algebraically equivalent to

$$\frac{x}{1+(nx)^2} \le \frac{1}{2n}.$$

Now we will apply this inequality. Let $\epsilon > 0$ be given, and choose $N > \frac{1}{2\epsilon}$. If $n \ge N$, then we have

$$\begin{aligned} |f_n(x) - 0| &= \left| \frac{x}{1+(nx)^2} - 0 \right| \\ &= \frac{x}{1+(nx)^2} \\ &\le \frac{1}{2n} \\ &\le \frac{1}{2N} \\ &< \epsilon. \end{aligned}$$

Therefore, $\{f_n\}$ converges uniformly to 0. □

(b) Show that $|f_n'(x)| \le 1$ for all x and all n.

Solution. We have the first derivative

$$\begin{aligned} f_n'(x) &= \frac{d}{dx} \frac{x}{1+(nx)^2} \\ &= \frac{1+(nx)^2 - 2(nx)^2}{(1+(nx)^2)^2} \\ &= \frac{1-(nx)^2}{(1+(nx)^2)^2}. \end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned} |f_n'(x)| &= \left| \frac{1-(nx)^2}{(1+(nx)^2)^2} \right| \\ &= \frac{|1-(nx)^2|}{(1+(nx)^2)^2} \\ &\le \frac{|1| + |-(nx)^2|}{(1+(nx)^2)^2} \\ &= \frac{1+(nx)^2}{(1+(nx)^2)^2} \\ &= \frac{1}{1+(nx)^2} \\ &\le 1. \end{aligned}$$

Therefore, we have established $|f_n'(x)| \le 1$ for all $x \in \mathbb{R}$ and all positive integers n. □

(c) Show that $\{f_n'\}$ converges pointwise to a function discontinuous at the origin.

Solution. We will show that the limit of $\{f_n'\}$ is

$$g(x) := \lim_{n\to\infty} f_n'(x) = \begin{cases} 0 & \text{if } x \ne 0, \\ 1 & \text{if } x = 0. \end{cases}$$

which is discontinuous at the origin. Suppose $x \ne 0$. Let $\epsilon > 0$ be given, and choose $N > \frac{1}{\sqrt{\epsilon}x}$. If $n \ge N$, then we have

$$\begin{aligned} |f_n'(x) - g(x)| &= |f_n'(x) - 0| \\ &= |f_n'(x)| \\ &\le \frac{1}{1+(nx)^2} \\ &< \frac{1}{(nx)^2} \\ &\le \frac{1}{(Nx)^2} \\ &< \epsilon. \end{aligned}$$

At $x = 0$; the argument becomes trivial; we have

$$\begin{aligned}|f_n'(0) - g(0)| &= \left|\frac{1}{(1+(n(0))^2)^2} - 1\right| \\ &= |1 - 1| \\ &= 0 \\ &< \epsilon.\end{aligned}$$

Therefore, $f_n'(x)$ converges pointwise to $g(x)$ for all $x \in \mathbb{R}$. □

3. (Exercise 5.4.2): Let $b > 0, b \neq 1$ be given.

(a) Show that for every $y > 0$, there exists a unique number x such that $y = b^x$. Define the logarithm base b, $\log_b : (0, \infty) \to \mathbb{R}$, by $\log_b(y) := x$

Solution (by Estela Gavosto). Notice that it suffices to show that, for every $y > 0$, there exists a unique number x such that $y = b^x$. That is, we have to show that $y = b^x = \exp(x \ln(b))$ is one-to-one. Suppose that, given $y > 0$, there exist x_1, x_2 such that $y = b^{x_1} = b^{x_2}$. Then we have

$$\begin{aligned}\ln(y) &= \ln(\exp(x_1 \ln(b)) \\ &= x_1 \ln(b),\end{aligned}$$

or equivalently $x_1 = \frac{\ln(y)}{\ln(b)}$. Similarly, we have

$$\begin{aligned}\ln(y) &= \ln(\exp(x_2 \ln(b)) \\ &= x_2 \ln(b),\end{aligned}$$

or equivalently $x_2 = \frac{\ln(y)}{\ln(b)}$. So we obtain $x_1 = x_2$, and so we conclude that $y = b^x$ is one-to-one. We are now able to define the logarithm base b, written $\log_b : (0, \infty) \to \mathbb{R}$, by $\log_b(y) := x$. □

(b) Show that $\log_b(x) = \frac{\ln(x)}{\ln(b)}$.

Solution. Let $y := \log_b(x)$. Then by part (a) we can write $x = b^y$. Now substitute $u := t^{\frac{1}{y}}$, which gives $\frac{y}{u}\, du = \frac{1}{t}\, dt$, and so we have

$$\begin{aligned}\ln(x) &= \ln(b^y) \\ &= \int_1^{b^y} \frac{1}{t}\, dt \\ &= \int_1^{b} \frac{y}{u}\, du \\ &= y \int_1^{b} \frac{1}{u}\, du \\ &= y \ln(b).\end{aligned}$$

So we conclude $y = \frac{\ln(x)}{\ln(b)}$, or $\log_b(x) = \frac{\ln(x)}{\ln(b)}$, as desired. □

(c) Prove that if $c > 0, c \neq 1$, then $\log_b(x) = \frac{\log_c(x)}{\log_c(b)}$.

Solution. Write $y := \log_b(x)$. Then by part (a), we can write $x = b^y$. By part (b) and our proof of part (b), we obtain

$$\begin{aligned}\log_c(x) &= \log_c(b^y) \\ &= \frac{\ln(b^y)}{\ln(c)} \\ &= \frac{y\ln(b)}{\ln(c)} \\ &= y\frac{\ln(b)}{\ln(c)} \\ &= y \log_c(b).\end{aligned}$$

So we conclude $y = \frac{\log_c(x)}{\log_c(b)}$, or $\log_b(x) = \frac{\log_c(x)}{\log_c(b)}$, as desired. □

(d) Prove $\log_b(xy) = \log_b(x) + \log_b(y)$, and $\log_b(x^y) = y \log_b(x)$

Solution. The professor has already proved $\ln(xy) = \ln(x) + \ln(y)$ in her lecture notes. Using this and part (b), we obtain

$$\begin{aligned}\log_b(xy) &= \frac{\ln(xy)}{\ln(b)} \\ &= \frac{\ln(x) + \ln(y)}{\ln(b)} \\ &= \frac{\ln(x)}{\ln(b)} + \frac{\ln(y)}{\ln(b)} \\ &= \log_b(x) + \log_b(y).\end{aligned}$$

Next, we have already proved $\ln(x^y) = y\ln(x)$ in our proof of part (b). Using this and part (b), we obtain

$$\begin{aligned}\log_b(x^y) &= \frac{\ln(x^y)}{\ln(b)} \\ &= \frac{y\ln(x)}{\ln(b)} \\ &= y\frac{\ln(x)}{\ln(b)} \\ &= y\log_b(x),\end{aligned}$$

as desired. □

4. (Exercise 5.4.9): Using the logarithm find

$$\lim_{n\to\infty} n^{\frac{1}{n}}$$

Note: If you want to use L'Hopital's rule, you need to prove the result first. Alternatively, you can use directly the mean value theorem.

Solution. I will prove this using the Mean Value Theorem. Since $\ln(x)$ is a continuous function for all $x > 0$, the Mean Value Theorem asserts that, for all integers $n \geq 2$, there exists $k_n \in [1, n]$ that satisfies

$$\frac{d}{dx}\ln(x)\bigg|_{x=k_n} = \frac{\ln(n) - \ln(1)}{n-1},$$

or equivalently

$$k_n = \frac{n-1}{\ln(n)}.$$

Next, we want to show $\ln(n) < \sqrt{n}$ for all integers $n \geq 2$. One way (that I am choosing) to prove our claim is to define $f : (0,\infty) \to \mathbb{R}$ by $f(x) := \sqrt{x} - \ln(x)$. Then we have

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left(\sqrt{x} - \ln(x)\right) \\ &= \frac{1}{2\sqrt{x}} - \frac{1}{x} \\ &= \frac{x-2}{\sqrt{x}} \\ &\geq 0\end{aligned}$$

for all $x \geq 2$, which means that f is increasing for all $x \geq 2$. We also have $f(2) = \sqrt{2} - \ln(2) > 0$. So we conclude $f(x) > 0$, or equivalently $\ln(x) < \sqrt{x}$, for all $x \geq 2$. In particular, for all integers $n \geq 2$, we have $\ln(n) < \sqrt{n}$. Therefore, we have

$$\begin{aligned}k_n &= \frac{n-1}{\ln(n)} \\ &> \frac{n-1}{\sqrt{n}} \\ &= \sqrt{n} - \frac{1}{\sqrt{n}},\end{aligned}$$

which implies $\lim\limits_{n\to\infty} k_n = \infty$, and in turn $\lim\limits_{n\to\infty} \frac{1}{k_n} = 0$. So we have

$$\begin{aligned}\ln(n^{\frac{1}{n}}) &= \frac{\ln(n)}{n} \\ &= \frac{n-1}{n}\frac{\ln(n)}{n-1} \\ &= \left(1 - \frac{1}{n}\right)\frac{1}{k_n}.\end{aligned}$$

Since $\ln(x)$ is continuous for all $x > 0$, we have

$$\begin{aligned}\lim_{n\to\infty} \ln(n^{\frac{1}{n}}) &= \lim_{n\to\infty}\left(1-\frac{1}{n}\right)\lim_{n\to\infty}\frac{1}{k_n}\\ &= 1\cdot 0\\ &= 0.\end{aligned}$$

Since e^x is continuous for all $x \in \mathbb{R}$, we have

$$\begin{aligned}\lim_{n\to\infty} n^{\frac{1}{n}} &= \lim_{n\to\infty} e^{\ln(n^{\frac{1}{n}})}\\ &= e^{\lim_{n\to\infty}\ln(n^{\frac{1}{n}})}\\ &= e^0\\ &= \boxed{1},\end{aligned}$$

as desired. □

5. (Exercise 5.4.4): Use the geometric sum formula to show (for $t \neq -1$)

$$1 - t + t^2 - \cdots + (-1)^n t^n = \frac{1}{1+t} - \frac{(-1)^{n+1}t^{n+1}}{1+t}$$

Using this fact show

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}$$

for all $x \in (-1, 1]$ (note that $x = 1$ is included). Finally, find the limit of the alternating harmonic series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Solution. Suppose $|t| < 1$. We have

$$\begin{aligned}1 - t + t^2 - \cdots + (-1)^n t^n &= 1 + (-t) + (-t)^2 - \cdots + (-t)^n\\ &= \frac{1-(-t)^{n+1}}{1-(-t)}\\ &= \frac{1-(-1)^{n+1}t^{n+1}}{1+t}\\ &= \frac{1}{1+t} - \frac{(-1)^{n+1}t^{n+1}}{1+t}.\end{aligned}$$

Next, we claim the sequence of terms $\{\frac{(-1)^{n+1}t^{n+1}}{1+t}\}$ of the summation converges uniformly to 0 for all $|t| < 1$, which will allow us to interchange the summation and integral signs in our final calculations. To this end, let $\epsilon > 0$ be given, and, if we assume $t \in (-1, 0) \cup (0, 1)$, choose $N > \log_{|t|}(\epsilon) - 1$. If $n \geq N$, then we have

$$\begin{aligned}\left|\frac{(-1)^{n+1}t^{n+1}}{1+t} - 0\right| &= \frac{|t|^{n+1}}{1+t}\\ &\leq \frac{|t|^{n+1}}{1}\\ &= |t|^{n+1}\\ &\leq |t|^{N+1}\\ &< \epsilon.\end{aligned}$$

Otherwise, if $t = 0$, then the argument becomes trivial; we have

$$\begin{aligned}\left|\frac{(-1)^{n+1}(0)^{n+1}}{1+0} - 0\right| &= |0 - 0|\\ &= 0\\ &< \epsilon.\end{aligned}$$

So we conclude that $\{\frac{(-1)^{n+1}t^{n+1}}{1+t}\}$ of the summation converges uniformly to 0 for all $|t| < 1$, as we claimed. Now, we have

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k t^k &= \lim_{n\to\infty}(1 - t + t^2 - \cdots + (-1)^n t^n)\\ &= \lim_{n\to\infty}\left(\frac{1}{1+t} - \frac{(-1)^{n+1}t^{n+1}}{1+t}\right)\\ &= \frac{1}{1+t}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\ln(1+x) &= \ln(1+x) - \ln(1+0)\\ &= \ln(1+t)|_0^x\\ &= \int_0^x \frac{1}{1+t}\,dt\\ &= \int_0^x \sum_{k=0}^{\infty}(-1)^k t^k\,dt\\ &= \sum_{k=0}^{\infty}(-1)^k \int_0^x t^k\,dt\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k t^{k+1}}{k+1}\bigg|_0^x\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k x^{k+1}}{k+1}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}\end{aligned}$$

for all $-1 < x < 1$. Since all the hypotheses of Abel's Theorem are true, Abel's Theorem asserts here that the above series also converges at $x = 1$. So we can substitute $x = 1$ to conclude

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \ln(2),$$

as desired. □

6. (Exercise 5.4.8): Show that e^x is convex, in other words, show that if $a \le x \le b$, then

$$e^x \le e^a\frac{b-x}{b-a} + e^b\frac{x-a}{b-a}.$$

Solution. Define $f : [a,b] \to \mathbb{R}$ by

$$f(x) := e^a\frac{b-x}{b-a} + e^b\frac{x-a}{b-a} - e^x.$$

The first derivative is

$$\begin{aligned}f'(x) &= e^a\frac{d}{dx}\frac{b-x}{b-a} + e^b\frac{d}{dx}\frac{x-a}{b-a} - \frac{d}{dx}e^x\\ &= e^a\frac{-1}{b-a} + e^b\frac{1}{b-a} - e^x\\ &= \frac{e^b - e^a}{b-a} - e^x.\end{aligned}$$

Setting $f'(x_0) = 0$ yields the positive critical point

$$x_0 = \ln\left(\frac{e^b - e^a}{b-a}\right)$$

Moreover, the second derivative is

$$\begin{aligned}f''(x) &= \frac{d}{dx}\left(\frac{e^b - e^a}{b-a} - e^x\right)\\ &= -e^x\\ &< 0,\end{aligned}$$

which implies that the critical point x_0 is a maximum. Finally, since we also have $f(a) = 0$ and $f(b) = 0$, we conclude $f(x) \geq 0$ for all $a \leq x \leq b$, which is equivalent to

$$e^x \leq e^a \frac{b-x}{b-a} + e^b \frac{x-a}{b-a}$$

for all $a \leq x \leq b$, as desired. □

Alternate solution. Define $f : [a, b] \to \mathbb{R}$ by

$$f(x) := e^a \frac{b-x}{b-a} + e^b \frac{x-a}{b-a} - e^x.$$

which is continuous (because it is the addition and scalar multiplication of the continuous functions $b - x, x - a, e^x$) and satisfies $f(a) = f(b) = 0$. We want to show $f(x) \geq 0$ for all $a \leq x \leq b$. By Rolle's Theorem (the special case of the Mean Value Theorem for zero slope), there exists $c \in [a, b]$ that satisfies $f'(c) = 0$. Additionally, we have the first derivative

$$\begin{aligned} f'(x) &= e^a \frac{d}{dx} \frac{b-x}{b-a} + e^b \frac{d}{dx} \frac{x-a}{b-a} - \frac{d}{dx} e^x \\ &= e^a \frac{-1}{b-a} + e^b \frac{1}{b-a} - e^x \\ &= \frac{e^b - e^a}{b-a} - e^x. \end{aligned}$$

First, suppose $a \leq x \leq c$. Since e^x is an increasing function of x, it follows that $x \leq c$ implies $e^x \leq e^c$, and so we have

$$\begin{aligned} f'(x) &= \frac{e^b - e^a}{b-a} - e^x \\ &\geq \frac{e^b - e^a}{b-a} - e^c \\ &= f'(c) \\ &= 0 \end{aligned}$$

for all $a \leq x \leq c$. So we have $f(a) = 0$ and f is increasing on $[a, c]$, which together imply $f(x) \geq 0$ for all $a \leq x \leq c$. Next, suppose $c \leq x \leq b$. Since e^x is an increasing function of x, it follows that $x \geq c$ implies $e^x \geq e^c$, and so we have

$$\begin{aligned} f'(x) &= \frac{e^b - e^a}{b-a} - e^x \\ &\leq \frac{e^b - e^a}{b-a} - e^c \\ &= f'(c) \\ &= 0 \end{aligned}$$

So we have $f(b) = 0$ and f is decreasing on $[c, b]$, which together imply $f(x) \geq 0$ for all $c \leq x \leq b$. Therefore, we conclude $f(x) \geq 0$ for all $a \leq x \leq b$, which is equivalent to

$$e^x \leq e^a \frac{b-x}{b-a} + e^b \frac{x-a}{b-a}$$

for all $a \leq x \leq b$, as desired. □