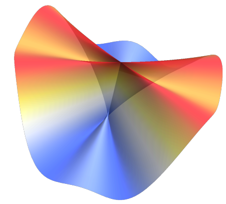
Department of Mathematics

MATH 151B - Advanced Calculus



Homework 4

Due date: Monday, February 8 at 11:59pm

Instructions:

* You are encourage to work in group but you have to write your individual solution
* You can use any book, article or web-based mathematical material or computational software
* You can ask Chulan, Ryan, or Estela questions
* Chegg, Math Stack Exchange, or any other source where you can copy solutions is not allowed
* The homework needs to be typeset in “LaTeX” .

1. (Exercise 7.1.7): Let X be the set of continuous functions on $[0,1]$. Let $\varphi\colon [0,1] \to (0,\infty)$ be continuous. Define
$$d(f,g) := \int_0^1 |f(x) - g(x)|\varphi(x)\, dx.$$
Show that (X,d) is a metric space.

2. (Exercise 7.1.12): Let $C^1([a,b],\mathbb{R})$ be the set of once continuously differentiable functions on $[a,b]$. Define

$$d(f,g) := \|f-g\|_u + \|f'-g'\|_u,$$

where $\|\cdot\|_u$ is the uniform norm. Prove that d is a metric.

3. (Exercise 7.2.9): Let X be a set and d_1, d_2 be two metrics on X. Suppose there exists an $\alpha > 0$ and $\beta > 0$ such that
$$\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_1(x,y) \text{ for all } x,y \in X.$$
Show that U is open in (X,d_1) if and only if U is open in (X,d_2). That is, the topologies of (X,d_1) and (X,d_2) are the same.

4. (Exercise 7.2.13): Let (X,d) be a metric space.

   a) For any $x \in X$ and $\delta > 0$, show $\overline{B(x,\delta)} \subset C(x,\delta)$.

   b) Is it always true that $\overline{B(x,\delta)} = C(x,\delta)$? Prove or find a counterexample.

5. (Exercise 7.2.18): For every $x \in \mathbb{R}^n$ and every $\delta > 0$ define the rectangle

$$R(x,\delta) := (x_1 - \delta, x_1 + \delta) \times (x_2 - \delta, x_2 + \delta) \times \cdots \times (x_n - \delta, x_n + \delta).$$

Show that these sets generate the same open sets as the balls in standard metric. That is, show that a set $U \subset \mathbb{R}^n$ is open in the sense of the standard metric if and only if for every point $x \in U$, there exists a $\delta > 0$ such that $R(x,\delta) \subset U$.

6. (Exercise 7.3.5): Suppose $\{x_n\}_{n=1}^{\infty}$ converges to x. Suppose $f\colon \mathbb{N} \to \mathbb{N}$ is a one-to-one function. Show that $\{x_{f(n)}\}_{n=1}^{\infty}$ converges to x.

7. (Exercise 7.3.7): A set $S \subset X$ is said to be *dense* in X if $X \subset \overline{S}$ or in other words if for every $x \in X$, there exists a sequence $\{x_n\}$ in S that converges to x. Prove that $\mathbb{R}^n$ contains a countable dense subset.