Homework 4 solutions

1. (Exercise 7.1.7): Let X be the set of continuous functions on $[0,1]$. Let $\varphi\colon [0,1] \to (0,\infty)$ be continuous. Define

$$d(f,g) := \int_0^1 |f(x) - g(x)|\varphi(x)\, dx.$$

Show that (X,d) is a metric space.

Proof. Let $f,g,h \in X$ be given. For nonnegativity, we have

$$\begin{aligned} d(f,g) &= \int_0^1 |f(x) - g(x)|\varphi(x)\, dx \\ &\geq \int_0^1 0\varphi(x)\, dx \\ &= 0 \end{aligned}$$

and that $d(f,g) = 0$ if and only if f,g satisfy

$$\int_0^1 |f(x) - g(x)|\varphi(x)\, dx = 0,$$

if and only if $|f(x) - g(x)| = 0$, if and only if $f(x) = g(x)$ for all $x \in [0,1]$, if and only if $f = g$. For symmetry, we have

$$\begin{aligned} d(f,g) &= \int_0^1 |f(x) - g(x)|\varphi(x)\, dx \\ &= \int_0^1 |g(x) - f(x)|\varphi(x)\, dx \\ &= d(g,f). \end{aligned}$$

For triangle inequality, we have

$$\begin{aligned} d(f,h) &= \int_0^1 |f(x) - h(x)|\varphi(x)\, dx \\ &= \int_0^1 |f(x) - g(x) + g(x) - h(x)|\varphi(x)\, dx \\ &\leq \int_0^1 (|f(x) - g(x)| + |g(x) - h(x)|)\varphi(x)\, dx \\ &= \int_0^1 |f(x) - g(x)|\varphi(x) + |g(x) - h(x)|\varphi(x)\, dx \\ &= \int_0^1 |f(x) - g(x)|\varphi(x)\, dx + \int_0^1 |g(x) - h(x)|\varphi(x)\, dx \\ &= d(f,g) + d(g,h). \end{aligned}$$

Therefore, d is a metric, and so (X,d) is a metric space. □

2. (Exercise 7.1.12): Let $C^1([a,b],\mathbb{R})$ be the set of once continuously differentiable functions on $[a,b]$. Define

$$d(f,g) := \|f - g\|_u + \|f' - g'\|_u,$$

where $\|\cdot\|_u$ is the uniform norm. Prove that d is a metric.

Proof. Let $f,g,h \in C^1([a,b],\mathbb{R})$ be given. For nonnegativity, we have

$$\begin{aligned} d(f,g) &= \|f - g\|_u + \|f' - g'\|_u \\ &\geq 0 + 0 \\ &= 0 \end{aligned}$$

and that $d(f,g) = 0$ if and only if f,g satisfy

$$\|f - g\|_u + \|f' - g'\|_u,$$

if and only if $\|f - g\|_u = 0$ and $\|f' - g'\|_u = 0$, if and only if $f(x) = g(x)$ and $f'(x) = g'(x)$ for all $x \in [0, 1]$, if and only if $f = g$. For symmetry, we have

$$\begin{aligned} d(f, g) &= \|f - g\|_u + \|f' - g'\|_u \\ &= \|g - f\|_u + \|g' - f'\|_u \\ &= d(g, f). \end{aligned}$$

For triangle inequality, we have

$$\begin{aligned} d(f, h) &= \|f - h\|_u + \|f' - h'\|_u \\ &= \|(f - g) + (g - h)\|_u + \|(f' - g') + (g' - h')\|_u \\ &\leq (\|f - g\|_u + \|g - h\|_u) + (\|f' - g'\|_u + \|g' - h'\|_u) \\ &= (\|f - g\|_u + \|f' - g'\|_u) + (\|g - h\|_u + \|g' - h'\|_u) \\ &= d(f, g) + d(g, h). \end{aligned}$$

Therefore, d is a metric. □

3. (Exercise 7.2.9): Let X be a set and d_1, d_2 be two metrics on X. Suppose there exists an $\alpha > 0$ and $\beta > 0$ such that

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y) \quad \text{for all } x, y \in X.$$

Show that U is open in (X, d_1) if and only if U is open in (X, d_2). That is, the topologies of (X, d_1) and (X, d_2) are the same.

Proof. First, suppose that U is open in (X, d_1). Consider a point $y \in B_2(x, \alpha\delta)$. Then we have

$$\begin{aligned} d_1(x, y) &\leq \frac{1}{\alpha} d_2(x, y) \\ &\leq \frac{1}{\alpha}(\alpha\delta) \\ &= \delta, \end{aligned}$$

which implies $y \in B_1(x, \delta)$. So we have $B_2(x, \alpha\delta) \subseteq B_1(x, \delta) \subset U$, which means that U is open in (X, d_2).
Conversely, suppose that U is open in (X, d_2). Consider a point $y \in B_1(x, \beta\delta)$. Then we have

$$\begin{aligned} d_2(x, y) &\leq \frac{1}{\beta} d_1(x, y) \\ &\leq \frac{1}{\beta}(\beta\delta) \\ &= \delta, \end{aligned}$$

which implies $y \in B_2(x, \delta)$. So we have $B_1(x, \alpha\delta) \subseteq B_2(x, \delta) \subset U$, which means that U is open in (X, d_1). □

4. (Exercise 7.2.13): Let (X, d) be a metric space.

(a) For any $x \in X$ and $\delta > 0$, show $\overline{B(x, \delta)} \subset C(x, \delta)$.

Proof. Suppose we have $x \in \overline{B(x, \delta)}$. Then x is in the intersection of all closed sets containing $B(x, \delta)$. In other words, x is in every closed set containing $B(x, \delta)$. For instance, $C(x, \delta)$ is the closed ball containing $B(x, \delta)$. Therefore, we have $x \in C(x, \delta)$, and so we conclude $\overline{B(x, \delta)} \subset C(x, \delta)$. □

(b) Is it always true that $\overline{B(x, \delta)} = C(x, \delta)$? Prove or find a counterexample.

Counterexample. Consider the discrete metric d defined by

$$d(x, y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y \end{cases}$$

for all $x, y \in (X, d)$. If we select $\delta := 1$, then the discrete metric d implies $B(x, 1) = \{y \in X : d(x, y) < 1\} = \{x\}$. Since singleton sets are closed, we have $\overline{B(x, 1)} = B(x, 1) = \{x\}$. But the closed ball is

$$\begin{aligned} C(x, 1) &= \{y \in X : d(x, y) \leq 1\} \\ &= B(x, 1) \cup \{y \in X : d(x, y) = 1\} \\ &= \{x\} \cup \{y\}. \end{aligned}$$

If x, y are distinct, then we conclude

$$\begin{aligned} \overline{B(x, 1)} &= \{x\} \\ &\neq \{x\} \cup \{y\} \\ &= C(x, 1), \end{aligned}$$

as desired. □

5. (Exercise 7.2.18): For every $x \in \mathbb{R}^n$ and every $\delta > 0$ define the rectangle

$$R(x, \delta) := (x_1 - \delta, x_1 + \delta) \times (x_2 - \delta, x_2 + \delta) \times \cdots \times (x_n - \delta, x_n + \delta).$$

Show that these sets generate the same open sets as the balls in standard metric. That is, show that a set $U \subset \mathbb{R}^n$ is open in the sense of the standard metric if and only if for every point $x \in U$, there exists a $\delta > 0$ such that $R(x, \delta) \subset U$.

Proof. Suppose $U \subset \mathbb{R}^n$ is open in the sense of the standard metric. Then for any $x = (x_1, x_2, \ldots, x_n) \in U$, there exists $\delta > 0$ that satisfies $B(x, n\delta) \subset U$. For any $y \in R(x, \delta)$, we have $|y_i - x_i| < \delta$ for all $i = 1, 2, \ldots, n$, which implies

$$\begin{aligned}
\|y - x\| &= \|(y_1, y_2, \ldots, y_n) - (x_1, x_2, \ldots, x_n)\| \\
&= \|(y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n)\| \\
&= \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2} \\
&\leq \sqrt{(y_1 - x_1)^2} + \sqrt{(y_2 - x_2)^2} + \cdots + \sqrt{(y_n - x_n)^2} \\
&= |y_1 - x_1| + |y_2 - x_2| + \cdots + |y_n - x_n| \\
&< \delta + \delta + \cdots + \delta \\
&= n\delta,
\end{aligned}$$

which signifies that the rectangle $R(x, \delta)$ is contained in $B(x, n\delta)$.

Conversely, suppose that, for every point $x \in U$, there exists a $\delta > 0$ such that $R(x, \delta) \subset U$. Consider the open ball $B(x, \frac{\delta}{n})$. Then for any $y \in B(x, \delta)$, we have

$$\begin{aligned}
|y_i - x_i| &\leq \|y - x\| \\
&= \delta,
\end{aligned}$$

which means $y \in R(x, \delta)$, and so $B(x, \frac{\delta}{n})$ is contained in the rectangle $R(x, \delta)$. So we have the set inclusions $B(x, \frac{\delta}{n}) \subset R(x, \delta) \subset U$, which signifies that $U \subset \mathbb{R}^n$ is open in the sense of the standard metric. □

6. (Exercise 7.3.5): Suppose $\{x_n\}_{n=1}^{\infty}$ converges to x. Suppose $f : \mathbb{N} \to \mathbb{N}$ is a one-to-one function. Show that $\{x_{f(n)}\}_{n=1}^{\infty}$ converges to x.

Proof (by Estela Gavosto). Let $\epsilon > 0$ be given. Since $\{x_n\}_{n=1}^{\infty}$ converges to x, there exists $N \in \mathbb{N}$ that satisfies $|x_n - x| < \epsilon$ for all integers $n \geq N$. Moreover, since f is a one-to-one function, we have $f(n) \geq N$ for all $n \geq M$, where

$$M := \max\{f^{-1}(\{1\}), f^{-1}(\{2\}), f^{-1}(\{3\}), \ldots, f^{-1}(\{N\}), 1\}$$

So we conclude $|x_{f(n)} - x| < \epsilon$ for all integers $n \geq M$, which means $\{x_{f(n)}\}_{n=1}^{\infty}$ converges to x. □

7. (Exercise 7.3.7): A set $S \subset X$ is said to be *dense* in X if $X \subset \overline{S}$ or in other words if for every $x \in X$, there exists a sequence $\{x_n\}$ in S that converges to x. Prove that $\mathbb{R}^n$ contains a countable dense subset.

Proof. Consider for instance the set $\mathbb{Q}^n := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \in \mathbb{Q}, i = 1, 2, \ldots, n\}$. Then we have $\mathbb{Q}^n \subset \mathbb{R}^n$, and we want to show that $\mathbb{Q}^n$ is countable and dense. First, we will now that ⋉ is dense. By Theorem 1.2.4(ii) of the Lebl textbook, $\mathbb{Q}$ is dense in $\mathbb{R}$, which means that, given any $i = 1, \ldots, n$, there exists a sequence $\{(x_i)_k\}_{k=1}^{\infty}$ in $\mathbb{Q}$ that converges to x_i. By definition, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, if $k \geq N$, then $|(x_i)_k - x_i| < \frac{\epsilon}{n}$. This implies

$$\begin{aligned}
\|(x_1, x_2, \ldots, x_n)_k - (x_1, x_2, \ldots, x_n)\| &= \|((x_1)_k, (x_2)_k, \ldots, (x_n)_k) - (x_1, x_2, \ldots, x_n)\| \\
&= \|((x_1)_k - x_1, (x_2)_k - x_2, \ldots, (x_n)_k - x_n)\| \\
&\leq |(x_1)_k - x_1| + |(x_2)_k - x_2| + \cdots + |(x_n)_k - x_n| \\
&< \frac{\epsilon}{n} + \frac{\epsilon}{n} + \cdots + \frac{\epsilon}{n} \\
&= \epsilon.
\end{aligned}$$

Therefore, $\{(x_1, x_2, \ldots, x_n)_k\}_{k=1}^{\infty}$ converges to $(x_1, x_2, \ldots, x_n)$, and so $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. Next, we will show that $\mathbb{Q}^n$ is also countable. I will prove this by induction. By Example 0.3.32 of the Lebl textbook, $\mathbb{Q}$ is countable. Now assume that $\mathbb{Q}^k$ is countable. We will prove that $\mathbb{Q}^{k+1}$ is countable. We can write $\mathbb{Q}^{k+1} = \mathbb{Q}^k \times \mathbb{Q} = \{((x_1, x_2, \ldots, x_k), y) \in \mathbb{R}^k \times \mathbb{R} : x_j \in \mathbb{R}^k, y \in \mathbb{R}, j = 1, \ldots, k\}$. Since we know that $\mathbb{Q}$ is countable and we assumed that $\mathbb{Q}^k$ is countable, the set $\mathbb{Q}^{k+1} = \mathbb{Q}^k \times \mathbb{Q}$ is in one-to-one correspondence with $\mathbb{N} \times \mathbb{N}$, which is countable by Example 0.3.31 of the Lebl textbook. So we conclude that $\mathbb{Q}^{k+1}$ is countable, completing our proof by induction. □