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Homework 5 solutions

1. (Exercise 7.4.14): Prove the general Bolzano-Weierstrass theorem: Any bounded sequence $\{x_k\}$ in $\mathbb{R}^n$ has a convergent subsequence.

   Solution. Since $\{x_k\}_{k=1}^{\infty}$ is a bounded sequence in $\mathbb{R}^n$, it follows that each $\{(x_i)_k\}_{k=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$ for any $i = 1, 2, \ldots, n$. By the Bolzano-Weierstrass Theorem for real numbers (Theorem 2.3.8 of Lebl), there exists a convergent subsequence $\{(x_i)_{k_l}\}_{l=1}^{\infty}$ in $\mathbb{R}$ whose limit is, say, $x_i \in \mathbb{R}$, for any $i = 1, 2, \ldots, n$. This implies by Proposition ??? that $\{x_{k_l}\}_{l=1}^{\infty}$ is a convergent subsequence in $\mathbb{R}^n$ whose limit is $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. □

2. (Exercise 7.5.6): Prove the following version of the intermediate value theorem. Let (X, d) be a connected metric space and $f : X \to \mathbb{R}$ a continuous function. Suppose that there exist $x_0, x_1 \in X$ and $y \in \mathbb{R}$ such that $f(x_0) < y < f(x_1)$. Then prove that there exists a $z \in X$ such that $f(z) = y$.

   Hint: See Exercise 7.5.5.

   Solution. Exercise 7.5.5 states that, if $f : X \to Y$ is continuous for metric spaces (X, d_X) and (Y, d_Y) and X is connected, then $f(X)$ is connected.

   Suppose we have $f(z) \neq y$ for all $z \in X$. Then we can write $f(X) = A \cup B$, where $A := f(X) \cap (-\infty, y)$ and $B := f(X) \cap (y, \infty)$. Notice that A, B are nonempty because we have $f(x_0) \in A$ and $f(x_1) \in B$. Also, A, B are disjoint because of

$$\begin{aligned} A \cap B &= (f(X) \cap (-\infty, y)) \cap (f(X) \cap (y, \infty)) \\ &= f(X) \cap ((-\infty, y) \cap (y, \infty)) \\ &= f(X) \cap \varnothing \\ &= \varnothing. \end{aligned}$$

   Therefore, $f(X) = A \cup B$ is disconnected. By the contrapositive of Exercise 7.5.5, X is disconnected, which contradicts the assumption that X is connected. □

3. (Exercise 7.5.9): Take the metric space of continuous functions $C([0, 1], \mathbb{R})$. Let $k : [0, 1] \times [0, 1] \to \mathbb{R}$ be a continuous function. Given $f \in C([0, 1], \mathbb{R})$ define

$$\varphi_f(x) := \int_0^1 k(x, y) f(y)\, dy$$

   (a) Show that $T(f) := \varphi_f$ defines a function $T : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$.

   Solution. We need to show that T is well-defined. Assume $f, g \in C([0, 1], \mathbb{R})$ represent the function; in other words, suppose $f = g$. Then we have $f(y) = g(y)$ for all $0 \leq y \leq 1$, and so we have

$$\begin{aligned} \varphi_f(x) &= \int_0^1 k(x, y) f(y)\, dy \\ &= \int_0^1 k(x, y) g(y)\, dy \\ &= \varphi_g(x) \end{aligned}$$

   for all $0 \leq x \leq 1$, which implies

$$\begin{aligned} T(f) &= \varphi_f \\ &= \varphi_g \\ &= T(g). \end{aligned}$$

   Therefore, T is well-defined. Now we need to show that T maps f into the codomain $C([0, 1], \mathbb{R})$; that is, we need to show $T(f)$ is continuous on $[0, 1]$. Let $\epsilon > 0$ be given, and choose $\delta :=$. For brevity, write $J := \sup_{x \in [0,1]} |f(y)|$. Since

k is continuous, we have $|k(x,y) - k(c,y)| < \frac{\epsilon}{J}$. If $x, c \in [0,1]$ with c fixed satisfy $|x - c| < \delta$, then we have

$$\begin{aligned}
|(T(f))(x) - (T(f))(c)| &= |\varphi_f(x) - \varphi_f(c)| \\
&= \left| \int_0^1 k(x,y) f(y)\, dy - \int_0^1 k(c,y) f(y)\, dy \right| \\
&= \left| \int_0^1 (k(x,y) - k(c,y)) f(y)\, dy \right| \\
&\leq \int_0^1 |k(x,y) - k(c,y)| |f(y)|\, dy \\
&\leq \int_0^1 |k(x,y) - k(c,y)| J\, dy \\
&< \int_0^1 \frac{\epsilon}{J} J\, dy \\
&= \epsilon,
\end{aligned}$$

and so $T(f)$ is continuous on $[0,1]$. □

(b) Show that T is continuous.

Solution. Let $\epsilon > 0$ be given. Since $k : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function and $[0,1] \times [0,1]$ is a compact set (because $[0,1] \times [0,1] \subset \mathbb{R}^2$ is closed and bounded, and the general Bolzano-Weierstrass theorem applies here), the maximum of k over $[0,1] \times [0,1]$ exists, and so we can define $M := \max_{(x,y) \in [0,1] \times [0,1]} |k(x,y)|$. Now consider $f_0 \in C([0,1], \mathbb{R})$. There exists $\delta > 0$ that satisfies $|f(y) - f_0(y)| < \frac{\epsilon}{2M}$ for all $y \in [0,1]$. So we have

$$\begin{aligned}
|\varphi_f(x) - \varphi_{f_0}(x)| &= \left| \int_0^1 k(x,y) f(y)\, dy - \int_0^1 k(x,y) f_0(y)\, dy \right| \\
&= \left| \int_0^1 k(x,y)(f(y) - f_0(y))\, dy \right| \\
&\leq \int_0^1 |k(x,y)| |f(y) - f_0(y)|\, dy \\
&\leq \int_0^1 M |f(y) - f_0(y)|\, dy \\
&< \int_0^1 M \frac{\epsilon}{2M}\, dy \\
&= \int_0^1 \frac{\epsilon}{2}\, dy \\
&= \frac{\epsilon}{2},
\end{aligned}$$

which implies

$$\begin{aligned}
\|T(f) - T(g)\|_u &= \|\varphi_f - \varphi_g\|_u \\
&= \sup_{x \in [0,1]} |\varphi_f(x) - \varphi_{f_0}(x)| \\
&\leq \sup_{x \in [0,1]} \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

Therefore, T is continuous. □

4. (Exercise 8.3.1): Suppose $\gamma : (-1,1) \to \mathbb{R}^n$ and $\alpha : (-1,1) \to \mathbb{R}^n$ be two differentiable curves such that $\gamma(0) = \alpha(0)$ and $\gamma'(0) = \alpha'(0)$. Suppose $F : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function. Show that

$$\left.\frac{d}{dt}\right|_{t=0} F(\gamma(t)) = \left.\frac{d}{dt}\right|_{t=0} F(\alpha(t))$$

Solution. We use the chain rule to obtain

$$\begin{aligned}
\frac{d}{dt} F(\gamma(t)) &= F'(\gamma(t))\gamma'(t), \\
\frac{d}{dt} F(\alpha(t)) &= F'(\gamma(t))\alpha'(t)
\end{aligned}$$

for all $-1 < t < 1$. At $t = 0$, we obtain

$$\left.\frac{d}{dt}\right|_{t=0} F(\gamma(t)) = F'(\gamma(t))\gamma'(t)|_{t=0}$$
$$= F'(\gamma(0))\gamma'(0)$$
$$= F'(\alpha(0))\alpha'(0)$$
$$= F'(\alpha(t))\alpha'(t)|_{t=0}$$
$$= \left.\frac{d}{dt}\right|_{t=0} F(\alpha(t)),$$

as desired. □

5. (Exercise 8.3.10): Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and suppose that whenever $x^2 + y^2 = 1$, then $f(x, y) = 0$. Prove that there exists at least one point (x_0, y_0) such that $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$.

   Solution. Since f is differentiable on $\mathbb{R}^2$, it follows by Proposition 8.3.5 of the Lebl textbook that f is continuous on $\mathbb{R}^2$. Since f and satisfies $f(x, y) = 0$ on the unit circle $x^2 + y^2 = 1$, it follows that f attains a relative maximum or a relative minimum on the disk $x^2 + y^2 < 1$. In other words, there exists at least one point (x_0, y_0) such that $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$, as desired. □