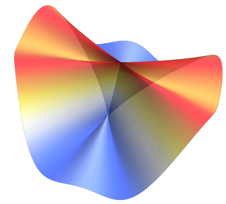
UC RIVERSIDE
Department of Mathematics

MATH 151B - Advanced Calculus



Homework 4

Due date: Tuesday, March 2 at 11:59pm

Instructions:

* You are encourage to work in group but you have to write your individual solution
* You can use any book, article or web-based mathematical material or computational software
* You can ask Chulan, Ryan, or Estela questions
* Chegg, Math Stack Exchange, or any other source where you can copy solutions is not allowed
* The homework needs to be typeset in “LaTeX” .

1. (Exercise 8.4.1): Define $f : \mathbb{R}^2 \to \mathbb{R}$ as:

$$f(x,y) := \begin{cases} (x^2+y^2)\sin\left((x^2+y^2)^{-1}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{else.} \end{cases}$$

Show that f is differentiable at the origin, but that it is not continuously differentiable.

Note: Feel free to use what you know about sine and cosine from calculus.

2. (Exercise 8.4.3): Let $B(0,1) \subset \mathbb{R}^2$ be the unit ball (disc), that is, the set given by $x^2+y^2<1$. Suppose $f : B(0,1) \to \mathbb{R}$ is a differentiable function such that $|f(0,0)| \leq 1$, and $\left|\frac{\partial f}{\partial x}\right| \leq 1$ and $\left|\frac{\partial f}{\partial y}\right| \leq 1$ for all points in $B(0,1)$.

   a) Find an $M \in \mathbb{R}$ such that $\|f'(x,y)\| \leq M$ for all $(x,y) \in B(0,1)$.
   b) Find a $B \in \mathbb{R}$ such that $|f(x,y)| \leq B$ for all $(x,y) \in B(0,1)$

3. (Exercise 8.4.8): Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ are two differentiable functions such that $f'(x) = h'(x)$ for all $x \in \mathbb{R}^n$. Prove that if $f(0) = h(0)$, then $f(x) = h(x)$ for all $x \in \mathbb{R}^n$.

4. (Exercise 8.5.3): Define $f : \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{(0,0)\}$ by $f(x,y) := (e^x \cos(y), e^x \sin(y))$.

   a) Show that f is onto.
   b) Show that f' is invertible at all points.
   c) Show that f is not one -to-one, in fact for every $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$, there exist infunitely many different points $(x,y) \in \mathbb{R}^2$ such that $f(x,y) = (a,b)$

   Therefore, invertible derivative at every point does not mean that f is invertible globally.

   Note: Feel free to use what you know about sine and cosine from calculus.

5. (Exercise 8.5.9): Let $H := \left\{(x,y) \in \mathbb{R}^2 : y > 0\right\}$, and for $(x,y) \in H$ define

$$F(x,y) := \left(\frac{x^2+y^2-1}{x^2+2y+y^2+1}, \frac{-2x}{x^2+2y+y^2+1}\right)$$

Prove that F is a bijective mapping from H to $B(0,1)$, it is contimuously differentiable on H, and its inverse is also continuously differentiable.

6. (Exercise 8.5.10): Suppose U $\subset \mathbb{R}^2$ is an open set and $f : U \to \mathbb{R}$ is a C^1 function such that $\nabla f(x,y) \neq 0$ for all $(x,y) \in U$. Show that every level set is aC^1 smooth curve. That is, for every $(x,y) \in U$, there exists aC^t function $\gamma : (-\delta, \delta) \to \mathbb{R}^2$ with $\gamma'(0) \neq 0$ such that $f(\gamma(t))$ is constant for all $t \in (-\delta, \delta)$.