Homework 6 solutions

1. (Exercise 8.4.1): Define $f : \mathbb{R}^2 \to \mathbb{R}$ as:

$$f(x,y) := \begin{cases} (x^2+y^2)\sin(\frac{1}{x^2+y^2}) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Show that f is differentiable at the origin, but that it is not continuously differentiable.

Note: Feel free to use what you know about sine and cosine from calculus.

Solution. According to Definition 8.3.1 of the Lebl textbook, f is differentiable at the origin $(0,0)$ if there exists $A \in L(\mathbb{R}^2, \mathbb{R})$ such that

$$\lim_{(h_1,h_2)\to(0,0)} \frac{\|f(h_1,h_2) - f(0,0) - A(h_1,h_2)\|}{\|(h_1,h_2)\|} = 0.$$

If the derivative exists, it is in $L(\mathbb{R}^2, \mathbb{R})$, so it can be represented by a 1×2 matrix $\begin{bmatrix} a & b \end{bmatrix}$ for some scalars $a, b \in \mathbb{R}$, which means

$$\begin{aligned} A(x,y) &= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= ax + by. \end{aligned}$$

So we have

$$\begin{aligned} \frac{\|f(h_1,h_2) - f(0,0) - A(h_1,h_2)\|}{\|(h_1,h_2)\|} &= \frac{\|(h_1^2+h_2^2)\sin(\frac{1}{h_1^2+h_2^2}) - 0 - (ah_1+bh_2)\|}{\|(h_1,h_2)\|} \\ &= \frac{\sqrt{(h_1^2+h_2^2)^2\sin^2(\frac{1}{h_1^2+h_2^2}) + (ah_1+bh_2)^2}}{\sqrt{h_1^2+h_2^2}} \\ &= \sqrt{(h_1^2+h_2^2)\sin^2\left(\frac{1}{h_1^2+h_2^2}\right) + \frac{(ah_1+bh_2)^2}{h_1^2+h_2^2}}. \end{aligned}$$

If we choose $a = 0$ and $b = 0$, so that the linear map becomes $A(x,y) = 0$, then our expression becomes

$$\begin{aligned} \frac{\|f(h_1,h_2) - f(0,0) - A(h_1,h_2)\|}{\|(h_1,h_2)\|} &= \sqrt{(h_1^2+h_2^2)\sin^2\left(\frac{1}{h_1^2+h_2^2}\right) + \frac{(0h_1+0h_2)^2}{h_1^2+h_2^2}} \\ &= \sqrt{h_1^2+h_2^2}\left|\sin\left(\frac{1}{h_1^2+h_2^2}\right)\right| \\ &\leq \sqrt{h_1^2+h_2^2}, \end{aligned}$$

which implies

$$\begin{aligned} \lim_{(h_1,h_2)\to(0,0)} \frac{\|f(h_1,h_2) - f(0,0) - A(h_1,h_2)\|}{\|(h_1,h_2)\|} &\leq \lim_{(h_1,h_2)\to(0,0)} \sqrt{h_1^2+h_2^2} \\ &= \sqrt{0^2+0^2} \\ &= 0, \end{aligned}$$

from which we conclude

$$\lim_{(h_1,h_2)\to(0,0)} \frac{\|f(h_1,h_2) - f(0,0) - A(h_1,h_2)\|}{\|(h_1,h_2)\|} = 0,$$

and so f is differentiable at the origin. Now we will show that f is not continuously differentiable. We have

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h_1\to 0} \frac{f(h_1,0) - f(0,0)}{h_1} \\ &= \lim_{h_1\to 0} \frac{(h_1^2+0^2)\sin(\frac{1}{h_1^2+0^2}) - 0}{h_1} \\ &= \lim_{h_1\to 0} h_1 \sin\left(\frac{1}{h_1^2}\right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial f}{\partial y}(0,0) &= \lim_{h_2\to 0} \frac{f(0,h_2)-f(0,0)}{h_2} \\
&= \lim_{h_2\to 0} \frac{(0^2+h_2^2)\sin(\frac{1}{0^2+h_2^2})-0}{h_2} \\
&= \lim_{h_2\to 0} h_2 \sin\left(\frac{1}{h_2^2}\right) \\
&= 0.
\end{aligned}$$

However, for all $(x, y) \neq (0, 0)$, we have the partial derivative

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= \frac{\partial}{\partial x}\left((x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)\right) \\
&= \frac{\partial}{\partial x}(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) + (x^2+y^2)\frac{\partial}{\partial x}\sin\left(\frac{1}{x^2+y^2}\right) \\
&= 2x\sin\left(\frac{1}{x^2+y^2}\right) + (x^2+y^2)\left(-\frac{2x}{(x^2+y^2)^2}\cos\left(\frac{1}{x^2+y^2}\right)\right) \\
&= 2x\sin\left(\frac{1}{x^2+y^2}\right) - \frac{2x}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right),
\end{aligned}$$

whose limit as $(x, y) \to (0, 0)$ does not exist because the limit

$$\lim_{(x,y)\to(0,0)} \frac{2x}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)$$

does not exist. Indeed, if we choose sequences $\{(x_n, y_n)\}$ and $\{(\tilde{x}_n, \tilde{y}_n)\}$ given by $x_n = y_n := \sqrt{\frac{1}{4\pi(1+n)}}$ and $\tilde{x}_n = \tilde{y}_n := \sqrt{\frac{1}{\pi(2+n)}}$, then we would have $x_n \to 0$ and $y_n \to 0$, but also

$$\begin{aligned}
\lim_{n\to\infty} \frac{2x_n}{x_n^2+y_n^2}\cos\left(\frac{1}{x_n^2+y_n^2}\right) &= \lim_{n\to\infty} \frac{2x_n}{x_n^2+y_n^2}\cos\left(\frac{1}{x_n^2+y_n^2}\right) \\
&= \lim_{n\to\infty} \sqrt{4\pi(1+n)}\cos(2\pi(1+n)) \\
&= \lim_{n\to\infty} \sqrt{4\pi(1+n)}\cdot 1 \\
&= \lim_{n\to\infty} \sqrt{4\pi(1+n)} \\
&= \infty
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} \frac{2\tilde{x}_n}{x_n^2+y_n^2}\cos\left(\frac{1}{x_n^2+y_n^2}\right) &= \lim_{n\to\infty} \frac{2x_n}{x_n^2+y_n^2}\cos\left(\frac{1}{x_n^2+y_n^2}\right) \\
&= \lim_{n\to\infty} \sqrt{2\pi(2+n)}\cos(\pi(2+n)) \\
&= \lim_{n\to\infty} \sqrt{2\pi(2+n)}\cdot(-1) \\
&= -\lim_{n\to\infty} \sqrt{2\pi(2+n)} \\
&= -\infty
\end{aligned}$$

This contradicts our earlier result $\frac{\partial f}{\partial x}(0,0) = 0$. So Proposition 8.4.6 implies here that f is not continuously differentiable. □

2. (Exercise 8.4.3): Let $B(0, 1) \subset \mathbb{R}^2$ be the unit ball (disc), that is, the set given by $x^2 + y^2 < 1$. Suppose $f : B(0, 1) \to \mathbb{R}$ is a differentiable function such that $|f(0, 0)| \leq 1$, and $|\frac{\partial f}{\partial x}| \leq 1$ and $|\frac{\partial f}{\partial y}| \leq 1$ for all points in $B(0, 1)$.

   a) Find an $M \in \mathbb{R}$ such that $\|f'(x, y)\| \leq M$ for all $(x, y) \in B(0, 1)$.

      Solution. We have for all $(x, y) \in B(0, 1)$ the derivative

$$f'(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix},$$

which implies its norm

$$\begin{aligned}\|f'(x,y)\| &= \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \\ &\leq \sqrt{\left(\frac{\partial f}{\partial x}\right)^2} + \sqrt{\left(\frac{\partial f}{\partial y}\right)^2} \\ &= \left|\frac{\partial f}{\partial x}\right| + \left|\frac{\partial f}{\partial y}\right| \\ &\leq 1 + 1 \\ &= 2 \\ &= M,\end{aligned}$$

provided that we choose $M := 2$. □

b) Find a $B \in \mathbb{R}$ such that $|f(x,y)| \leq B$ for all $(x,y) \in B(0,1)$

Solution. Since $B(0,1)$ is a convex open set and $f : B(0,1) \to \mathbb{R}$ is a differentiable function that satisfies $\|f'(x,y)\| \leq M$, by Proposition 8.4.2 of the Lebl textbook, we have

$$|f(x,y) - f(0,0)| \leq M\|(x,y) - (0,0)\|$$

for all $(x,y) \in B(0,1)$. We also have, using the reverse triangle inequality and $|f(0,0)| \leq 1$,

$$\begin{aligned}|f(x,y) - f(0,0)| &\geq |f(x,y)| - |f(0,0)| \\ &\geq |f(x,y)| - 1.\end{aligned}$$

Therefore, we conclude

$$\begin{aligned}|f(x,y)| &\leq |f(x,y) - f(0,0)| + 1 \\ &\leq M\|(x,y) - (0,0)\| + 1 \\ &= 2\|(x,y)\| + 1 \\ &= 2\sqrt{x^2 + y^2} + 1 \\ &< 2\sqrt{1} + 1 \\ &= 3 \\ &= B,\end{aligned}$$

provided that we choose $B := 3$. □

3. (Exercise 8.4.8): Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ are two differentiable functions such that $f'(x) = h'(x)$ for all $x \in \mathbb{R}^n$. Prove that if $f(0) = h(0)$, then $f(x) = h(x)$ for all $x \in \mathbb{R}^n$.

Solution. Define $g : \mathbb{R}^n \to \mathbb{R}$ by $g(x) := f(x) - h(x)$ for all $x \in \mathbb{R}^n$. Then, by Proposition 8.3.6 of the Lebl textbook, $g = f - h$ is differentiable, and we have the derivative $g'(x) = f'(x) - h'(x)$. In fact, with the assumption $f'(x) = h'(x)$ for all $x \in \mathbb{R}^n$, we have

$$\begin{aligned}g'(x) &= f'(x) - h'(x) \\ &= h'(x) - h'(x) \\ &= 0,\end{aligned}$$

which implies by Corollary 8.4.4 of the Lebl textbook that g is a constant. But the assumption $f(0) = g(0)$ implies

$$\begin{aligned}g(0) &= f(0) - h(0) \\ &= h(0) - h(0) \\ &= 0,\end{aligned}$$

which forces the constant to be zero. In other words, we have $g(x) = 0$ for all $x \in \mathbb{R}^n$, which is equivalent to the desired conclusion $f(x) - h(x) = 0$, or $f(x) = h(x)$, for all $x \in \mathbb{R}^n$. □

4. (Exercise 8.5.3): Define $f : \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{(0,0)\}$ by $f(x,y) := (e^x \cos(y), e^x \sin(y))$.

   (a) Show that f is onto.

Solution. First, we have

$$\begin{aligned}\|f(x,y)\| &= \sqrt{(e^x\cos(y))^2+(e^x\sin(y))^2}\\ &= \sqrt{e^{2x}(\cos^2(y)+\sin^2(y))}\\ &= \sqrt{e^{2x}}\\ &= |e^x|\\ &= e^x\\ &> 0,\end{aligned}$$

meaning that we have $f(x,y)\neq(0,0)$ for all $(x,y)\in\mathbb{R}^2$ and that f is onto for $\mathbb{R}^2\setminus\{(0,0)\}$. □

(b) Show that f' is invertible at all points.

Solution. For all $(x,y)\in\mathbb{R}^2$, we have

$$\begin{aligned}f'(x,y) &= \begin{bmatrix}\frac{\partial}{\partial x}(e^x\cos(y)) & \frac{\partial}{\partial y}(e^x\cos(y))\\ \frac{\partial}{\partial x}(e^x\sin(y)) & \frac{\partial}{\partial y}(e^x\sin(y))\end{bmatrix}\\ &= \begin{bmatrix}e^x\cos(y) & -e^x\sin(y)\\ e^x\sin(y) & e^x\cos(y)\end{bmatrix},\end{aligned}$$

whose Jacobian is

$$\begin{aligned}J_{f'}(x,y) &= \det(f'(x,y))\\ &= (e^x\cos(y))(e^x\cos(y))-(e^x\sin(y))(-e^x\sin(y))\\ &= e^{2x}(\cos^2(y)+\sin^2(y))\\ &= e^{2x}\\ &\neq 0,\end{aligned}$$

meaning that f' is invertible on $\mathbb{R}^2$. Let $(a,b)\in\mathbb{R}^2\setminus\{(0,0)\}$. Then we have the system of equations

$$\begin{aligned}a &= e^x\cos(y),\\ b &= e^x\sin(y),\end{aligned}$$

from which we can solve simultaneously to obtain

$$\begin{aligned}x &= \ln(\sqrt{a^2+b^2}),\\ y &= \cos^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right)=\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right),\end{aligned}$$

meaning that f is onto. □

(c) Show that f is not one-to-one, in fact for every $(a,b)\in\mathbb{R}^2\setminus\{(0,0)\}$, there exist infinitely many different points $(x,y)\in\mathbb{R}^2$ such that $f(x,y)=(a,b)$.

Solution. Since we have

$$\begin{aligned}-1 \leq \frac{a}{\sqrt{a^2+b^2}} \leq 1,\\ -1 \leq \frac{b}{\sqrt{a^2+b^2}} \leq 1,\end{aligned}$$

there exist $\theta_1,\theta_2\in[0,2\pi)$ that satisfy

$$\begin{aligned}\theta_1 &= \cos^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right),\\ \theta_2 &= \sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right),\end{aligned}$$

respectively. Furthermore, we can write

$$y=\theta_1+2\pi k=\theta_2+2\pi k$$

for any $k\in\mathbb{Z}$, which implies that there exist infinitely many different points $(x,y)\in\mathbb{R}^2$ such that $f(x,y)=(a,b)$. □

Therefore, invertible derivative at every point does not mean that f is invertible globally.

Note: Feel free to use what you know about sine and cosine from calculus.

5. (Exercise 8.5.9): Let $H := \{(x, y) \in \mathbb{R}^2 : y > 0\}$, and for $(x, y) \in H$ define

$$F(x, y) := \left(\frac{x^2 + y^2 - 1}{x^2 + 2y + y^2 + 1}, -\frac{2x}{x^2 + 2y + y^2 + 1}\right)$$

Prove that F is a bijective mapping from H to $B(0, 1)$, it is continuously differentiable on H, and its inverse is also continuously differentiable.

Solution. First, we will show that F maps H to $B(0, 1)$ because its norm satisfies. If we assume $(x, y) \in H$, then we have $y > 0$, and so we obtain

$$\begin{aligned}
\|F(x, y)\| &= \sqrt{\left(\frac{x^2 + y^2 - 1}{x^2 + 2y + y^2 + 1}\right)^2 + \left(-\frac{2x}{x^2 + 2y + y^2 + 1}\right)^2} \\
&= \frac{\sqrt{(x^2 + y^2 - 1)^2 + (-2x)^2}}{x^2 + 2y + y^2 + 1} \\
&= \frac{\sqrt{x^4 + 2x^2(y^2 - 1) + (y^2 - 1)^2 + 4x^2}}{x^2 + 2y + y^2 + 1} \\
&= \frac{\sqrt{x^4 + 2x^2y^2 - 2x^2 + y^4 - 2y^2 + 1 + 4x^2}}{x^2 + 2y + y^2 + 1} \\
&= \frac{\sqrt{x^4 + 2x^2y^2 + y^4 + 2x^2 - 2y^2 + 1}}{x^2 + 2y + y^2 + 1} \\
&< \frac{\sqrt{x^4 + 2x^2y^2 + y^4 + 2x^2 + 2y^2 + 1}}{x^2 + 2y + y^2 + 1} \\
&= \frac{\sqrt{(x^2 + y^2)^2 + 2(x^2 + y^2) + 1}}{x^2 + 2y + y^2 + 1} \\
&= \frac{\sqrt{(x^2 + y^2 + 1)^2}}{x^2 + 2y + y^2 + 1} \\
&= \frac{x^2 + y^2 + 1}{x^2 + 2y + y^2 + 1} \\
&< \frac{x^2 + y^2 + 1}{x^2 + y^2 + 1} \\
&= 1,
\end{aligned}$$

which implies that F maps into $B(0, 1)$. Next, we need to show that F is onto. Let $(a, b) \in B(0, 1)$. Then we can write

$$\begin{aligned}
a &= \frac{x^2 + y^2 - 1}{x^2 + 2y + y^2 + 1} = \frac{x^2 + y^2 - 1}{x^2 + (y + 1)^2}, \\
b &= -\frac{2x}{x^2 + 2y + y^2 + 1} = -\frac{2x}{x^2 + (y + 1)^2}.
\end{aligned}$$

Substitute $u := y + 1$, so that we can in fact write

$$\begin{aligned}
a &= \frac{x^2 + (u - 1)^2 - 1}{x^2 + u^2} = \frac{x^2 + u^2 - 2u}{x^2 + u^2} = 1 - \frac{2u}{x^2 + u^2}, \\
b &= -\frac{2x}{x^2 + u^2}.
\end{aligned}$$

Observe that we have

$$\begin{aligned}
(1 - a)^2 + b^2 &= \left(\frac{2u}{x^2 + u^2}\right)^2 + \left(-\frac{2x}{x^2 + u^2}\right)^2 \\
&= \frac{4u^2}{(x^2 + u^2)^2} + \frac{4x^2}{(x^2 + u^2)^2} \\
&= \frac{4(x^2 + u^2)}{(x^2 + u^2)^2} \\
&= \frac{4}{x^2 + u^2} \\
&= \frac{4}{x^2 + (y + 1)^2},
\end{aligned}$$

or equivalently

$$x^2 + (y+1)^2 = \frac{4}{(1-a)^2 + b^2}.$$

Also, from the second equation $b = -\frac{2x}{x^2+u^2}$, we obtain

$$\begin{aligned} x &= -\frac{x^2+u^2}{2} b \\ &= -\frac{\frac{4}{(1-a)^2+b^2}}{2} b \\ &= -\frac{2b}{(1-a)^2+b^2}. \end{aligned}$$

Likewise, the first equation $a = 1 - \frac{2u}{x^2+u^2}$ implies

$$\begin{aligned} y &= u - 1 \\ &= \frac{1-a}{2(x^2+u^2)} - 1 \\ &= \frac{(1-a)(x^2+u^2)}{2} - 1 \\ &= \frac{(1-a)(x^2+(y+1)^2)}{2} - 1 \\ &= \frac{1-a}{2} \frac{4}{(1-a)^2+b^2} - 1 \\ &= \frac{2(1-a)}{(1-a)^2+b^2} - 1 \\ &= \frac{2(1-a)}{(1-a)^2+b^2} - \frac{(1-a)^2+b^2}{(1-a)^2+b^2} \\ &= \frac{(1-a)(2-(1-a))+b^2}{(1-a)^2+b^2} \\ &= \frac{(1-a)(1+a)+b^2}{(1-a)^2+b^2} \\ &= \frac{1-a^2+b^2}{(1-a)^2+b^2} \\ &> 0 \end{aligned}$$

since $(a,b) \in B(0,1)$ (that is, $\|(a,b)\| < 1$ which implies $a < 1, b < 1$) implies $1 - a^2 + b^2 \geq 1 - a^2 > 0$. In other words, we found $(x,y) \in H$ as an explicit expression of $(a,b) \in B(0,1)$; that is,

$$(x,y) = \left(-\frac{2b}{(1-a)^2+b^2}, \frac{1-a^2+b^2}{(1-a)^2+b^2}\right),$$

which implies that F is onto. Next, we need to show that F is one-to-one; we will establish: For all $(x_1,y_1),(x_2,y_2) \in H$ and $(a_1,b_1),(a_2,b_2) \in B(0,1)$, if $(x_1,y_1) \neq (x_2,y_2)$, then $(a_1,b_1) \neq (a_2,b_2)$. Suppose $(x_1,y_1),(x_2,y_2) \in H$ are distinct; that is, we have $(x_1,y_1) \neq (x_2,y_2)$. Then this is equivalent to

$$\left(-\frac{2b_1}{(1-a_1)^2+b_1^2}, \frac{1-a_1^2+b_1^2}{(1-a_1)^2+b_1^2}\right) \neq \left(-\frac{2b_2}{(1-a_2)^2+b_2^2}, \frac{1-a_2^2+b_2^2}{(1-a_2)^2+b_2^2}\right),$$

which is in turn equivalent to

$$(-2b_1, 1-a_1^2+b_1^2) \neq (-2b_2, 1-a_2^2+b_2^2)$$

for any $(a_1,b_1),(a_2,b_2) \in B(0,1)$. Coordinate-wise, we have *one* of the following:

$$\begin{gathered} -2b_1 \neq -2b_2, \\ 1-a_1^2+b_1^2 \neq 1-a_2^2+b_2^2. \end{gathered}$$

If we assume $-2b_1 \neq -2b_2$, then we obtain $b_1 \neq b_2$ and therefore $(a_1,b_1) \neq (a_2,b_2)$, and so we are done with this case. If we assume $1-a_1^2+b_1^2 \neq 1-a_2^2+b_2^2$, then we have equivalently $a_1^2 - a_2^2 \neq b_1^2 - b_2^2$, which is in turn equivalent to

$$(a_1+a_2)(a_1-a_2) \neq (b_1+b_2)(b_1-b_2).$$

If we assume $a_1 \neq a_2$ and $b_1 \neq b_2$, then we already have $(a_1,b_1) \neq (a_2,b_2)$. If we assume $a_1 = a_2$, then $a_1 - a_2 = 0$, which implies $b_1 + b_2 \neq 0$ and $b_1 - b_2 \neq 0$; in particular, we have $b_1 - b_2 \neq 0$, which is equivalent to $b_1 \neq b_2$, which implies

$(a_1, b_1) = (a_2, b_1) \neq (a_2, b_2)$. By similar reasoning, if we assume $b_1 = b_2$, then we have $(a_1, b_1) = (a_1, b_2) \neq (a_2, b_2)$. Finally, notice that the remaining case $a_1 = a_2$ and $b_1 = b_2$ presents an immediate contradiction to $(a_1 + a_2)(a_1 - a_2) \neq (b_1 + b_2)(b_1 - b_2)$. Therefore, F is ont-to-one. Since F is both one-to-one and onto, we conclude that F is bijective. We also have the derivative

$$\begin{aligned} F'(x,y) &= \begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{x^2+y^2-1}{x^2+2y+y^2+1}\right) & \frac{\partial}{\partial y}\left(\frac{x^2+y^2-1}{x^2+2y+y^2+1}\right) \\ \frac{\partial}{\partial x}\left(-\frac{2x}{x^2+2y+y^2+1}\right) & \frac{\partial}{\partial y}\left(-\frac{2x}{x^2+2y+y^2+1}\right) \end{bmatrix} \\ &= \begin{bmatrix} \frac{4x(y+1)}{(x^2+2y+y^2+1)^2} & \frac{2(y^2-x^2+2y+1)}{(x^2+2y+y^2+1)^2} \\ -\frac{2(y^2-x^2+2y+1)}{(x^2+2y+y^2+1)^2} & \frac{4x(y+1)}{(x^2+2y+y^2+1)^2} \end{bmatrix}, \end{aligned}$$

which implies the Jacobian

$$\begin{aligned} J_{F'}(x,y) &= \det(F'(x,y)) \\ &= \frac{4x(y+1)}{(x^2+2y+y^2+1)^2}\frac{4x(y+1)}{(x^2+2y+y^2+1)^2} - \left(-\frac{2(y^2-x^2+2y+1)}{(x^2+2y+y^2+1)^2}\right)\frac{2(y^2-x^2+2y+1)}{(x^2+2y+y^2+1)^2} \\ &= \frac{16x^2(y+1)^2 + 4(y^2-x^2+2y+1)^2}{(x^2+2y+y^2+1)^4}. \end{aligned}$$

If $x \neq 0$, then we have

$$\begin{aligned} J_{F'}(x,y) &= \frac{16x^2(y+1)^2 + 4(y^2-x^2+2y+1)^2}{(x^2+2y+y^2+1)^4} \\ &\geq \frac{16x^2(y+1)^2}{(x^2+2y+y^2+1)^4} \\ &\geq \frac{16x^2}{(x^2+2y+y^2+1)^4} \\ &> 0, \end{aligned}$$

and if $x = 0$, then we have

$$\begin{aligned} J_{F'}(x,y) &= \frac{16x^2(y+1)^2 + 4(y^2-x^2+2y+1)^2}{(x^2+2y+y^2+1)^4} \\ &= \frac{16(0)^2(y+1)^2 + 4(y^2-(0)^2+2y+1)^2}{((0)^2+2y+y^2+1)^4} \\ &= \frac{4(y^2+2y+1)^2}{(y^2+2y+1)^4} \\ &= \frac{4}{(y^2+2y+1)^2} \\ &> 0. \end{aligned}$$

In either case, we have $J_{F'}(x,y) > 0$ (namely, $J_{F'}(x,y) \neq 0$), which implies that F' is invertible, from which the Inverse Function Theorem implies that F is injective on H. Since F is both locally one-to-one and onto, F is bijective. Also, since all the matrix entries of $F'(x,y)$ are continuous, Proposition 8.4.6 of the Lebl textbook asserts that F is continuously differentiable. Also by the Inverse Function Theorem, there exists locally a function $g : B(0,1) \to H$ defined by $g(u,v) := F^{-1}(u,v)$ for all $(u,v) \in B(0,1)$ that is continuously differentiable. I am not sure about writing an argument about proving the global inverse. □

6. (Exercise 8.5.10): Suppose $U \subset \mathbb{R}^2$ is an open set and $f : U \to \mathbb{R}$ is a C^1 function such that $\nabla f(x,y) \neq 0$ for all $(x,y) \in U$. Show that every level set is a C^1 smooth curve. That is, for every $(x,y) \in U$, there exists a C^1 function $\gamma : (-\delta,\delta) \to \mathbb{R}^2$ with $\gamma'(0) \neq 0$ such that $f(\gamma(t))$ is constant for all $t \in (-\delta,\delta)$.

Solution. Since we have $\nabla f \neq 0$, it follows from the definition of the gradient $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ that we have either $\frac{\partial f}{\partial x} \neq 0$ or $\frac{\partial f}{\partial y} \neq 0$. Assume $\frac{\partial f}{\partial y} \neq 0$ without loss of generality. Now, fix a point $(x_0,y_0) \in U$ and define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x,y) := f(x,y) - f(x_0,y_0).$$

Then we have $F(x_0,y_0) = f(x_0,y_0) - f(x_0,y_0) = 0$ and $\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} \neq 0$; namely, $F'(x,y) \neq 0$. By the Implicit Function Theorem, there exist open sets $W, W' \subset \mathbb{R}$ such that $x_0 \in W$, $y_0 \in W'$, $W \times W' \subset \mathbb{R}^2$ and a $C^1(W)$ map $y : W \to W'$ with $y(x_0) = y_0$, and for all $(x,y) \in W$ such that $F(x,y(x)) = 0$. Since $W \subset \mathbb{R}$ is an open set, there exists $\delta > 0$ such that $B(x_0,\delta) \subset W$. Now we can define $\gamma : (-\delta,\delta) \to \mathbb{R}^2$ by

$$\gamma(t) := (x_0 + t, y(x_0 + t)).$$

(Notice that we have $\gamma(t) \in W \times W'$ for all $-\delta < t < \delta$, which means $F(x_0 + t, y(x_0 + t)) = 0$.) Then we have

$$\begin{aligned}\gamma(0) &= (x_0 + 0, y(x_0 + 0)) \\ &= (x_0, y(x_0)) \\ &= (x_0, y_0).\end{aligned}$$

We also obtain its first derivative

$$\begin{aligned}\gamma'(t) &= \frac{d}{dt}(x_0 + t, y(x_0 + t)) \\ &= \left(\frac{d}{dt}(x_0 + t), \frac{d}{dt}(y(x_0 + t))\right) \\ &= (1, y'(x_0 + t)),\end{aligned}$$

from which we see in particular

$$\begin{aligned}\gamma'(0) &= (1, y'(x_0 + 0)) \\ &= (1, y'(x_0)) \\ &\neq (0, 0)\end{aligned}$$

because we have, of course, $1 \neq 0$ in the first coordinate. Finally, for all $t \in (-\delta, \delta)$, we have

$$\begin{aligned}f(\gamma(t)) &= F(\gamma(t)) + f(x_0, y_0) \\ &= F(x_0 + t, y(x_0 + t)) + f(x_0, y_0) \\ &= 0 + f(x_0, y_0) \\ &= f(x_0, y_0) \\ &= f(x_0 + 0, y(x_0 + 0)) \\ &= f(\gamma(0)),\end{aligned}$$

which implies that $f(\gamma(t))$ is constant for all $t \in (-\delta, \delta)$. □