Definition:

Let (X,d) be a metric space.
A sequence $\{x_n\}$ is a Cauchy Sequence if, for every $\varepsilon>0$, there exists $M\in\mathbb{N}$ s.t. if $n\geq M$ and $k\geq M$, we have $d(x_n, x_k)<\varepsilon$

"$|x_n-x_k|<\varepsilon$"

Definition: Let (X,d) be a metric space we say that X is complete if every Cauchy sequence $\{x_n\}$ converges to a point $x\in X$

Example: Let $C([a,b],\mathbb{R})$ be the set of real-valued functions on $[a,b]$. Define d by $d(f,g)=\sup\limits_{x\in[a,b]}|f(x)-g(x)|$

Show that $C([a,b],\mathbb{R})$ is a complete metric space.

Proof: first show d is a metric, consider $f,g,h\in C([a,b],\mathbb{R})$

Non-negativity

$$\begin{aligned} d(f,g) &= \sup|f(x)-g(x)| \\ &\geq |f(x)-g(x)| \\ &\geq 0 \end{aligned}$$

$$\left.\begin{aligned} d(f,g)=0 &\Leftrightarrow \sup|f(x)-g(x)|=0 \\ &\Leftrightarrow |f(x)-g(x)|=0 \\ &\Leftrightarrow f(x)-g(x)=0 \\ & \quad f(x)=g(x) \end{aligned}\right\} \text{for all } x\in[a,b]$$

Symmetry

$$\begin{aligned} d(f,g) &= \sup|f(x)-g(x)|: x\in D \\ &= \sup|g(x)-f(x)|: x\in D \\ &= d(g,f) \end{aligned}$$

triangle inequality

$$\begin{aligned} d(f,h) &= \sup|f-h| \\ &= \sup|f-g+g-h| \\ &= \sup(|f-g|+|g-h|) \\ &\leq \sup(|f-g|)+\sup(|g-h|) \\ &= d(f,g)+d(g,h) \end{aligned}$$

$\Rightarrow d$ is a metric, and so $C([a,b],\mathbb{R})$ is a metric space

Now we will show that $C([a,b],\mathbb{R})$ is complete

Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $C([a,b],\mathbb{R})$
We want to show that $\{f_n\}_{n=1}^{\infty}$ converges to its limit, say f.
Since $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence of Functions, $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence of numbers for all $x\in[a,b]$

By Prop 2.4.4 $\underbrace{\{f_n(x)\}_{n=1}^{\infty}}_{a_n}$ is also bounded
By the Bolzano-Werestrass Thereom (2.38)
There exists a convergent sequence $\underbrace{\{f_{n_i}(x)\}_{i=1}^{\infty}}_{a_{n_i}}$ converges to its limit $f(x)$

Therefore,

$$\begin{aligned} |f_n(x)-f(x)| &= |f_n(x)-f_{n_i}(x)+f_{n_i}(x)-f_{n_k}(x)+f_{n_k}(x)-f(x)| \\ &\leq \underbrace{|f_n(x)-f_{n_i}(x)|}_{\text{Since } \{f_{n_i}(x)\} \text{ is convergent}}+\underbrace{|f_{n_i}(x)-f_{n_k}(x)|}_{\substack{\text{Since } f_n(x) \text{ is Cauchy} \\ (\Rightarrow \{f_{n_k}(x)\} \text{ is Cauchy})}}+\underbrace{|f_{n_k}(x)-f(x)|}_{\text{Since } \{f_{n_i}(x)\} \text{ is convergent}} \\ &< \varepsilon/3+\varepsilon/3+\varepsilon/3 \\ &= \varepsilon \end{aligned}$$

So $\{f_n(x)\}_{n=1}^{\infty}$ converges to $f(x)$ for all $x\in[a,b]$
Therefore, $\{f_n\}_{n=1}^{\infty}$ converges to f in $C([a,b],\mathbb{R})$

Example: Suppose $f: X \to Y$ is continuous for metric spaces (X, d_X) and (Y, d_Y). Show that if X is connected, then $f(X)$ is connected.

Pf. We will prove by contrapositive

If $f(X)$ is disconnected, then X is disconnected

Since we assume $f(X)$ is disconnected we can write $f(X) = A \cup B$, with A, B nonempty and $A \cap B = \emptyset$

Let U_a be an open neighborhood of $f(a) \in A$, and let U_b be an open neighborhood of $f(u) \in B$, s.t. $U_a \subseteq A$, $U_b \subseteq B$ and $U_a \cup U_b = Y$

Since $f: X \to Y$ is continuous, by Lemma 7.5.7, the set $f^{-1}(U_a)$ contains a nonempty, open neighborhood W_a of $a \in X$, and the set $f^{-1}(U_b)$ contains a nonempty, open neighborhood W_b of $b \in X$

Since $W_a \subset f^{-1}(U_a)$ is open
$W_b \subset f^{-1}(U_b)$ is open
it follows that (respectively) $f^{-1}(U_a)$, $f^{-1}(U_b)$ are open

Furthermore, we have

prove this using basic set theory & functions

$$f^{-1}(U_a) \cup f^{-1}(U_b) = f^{-1}(U_a \cup U_b)$$
$$= f^{-1}(Y)$$
$$= X$$

and

$$f^{-1}(U_a) \cap f^{-1}(U_b) = f^{-1}(U_a \cap U_b) = f^{-1}(\emptyset) = \emptyset$$

So X is a disjoint union of $f^{-1}(U_a)$ and $f^{-1}(U_b)$

Namely, X is disconnected.