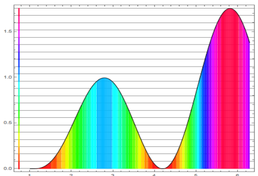
UC RIVERSIDE
Department of Mathematics

MATH 151C - Advanced Calculus



HIGHER STAKES HOMEWORK 1

Due date: Wednesday, March 21 at 11:59pm

Instructions:

- Work individually in the problems. You can ask questions to Estela, Chulan, or Ryan
- You can use any book, article or web-based mathematical material or computational software
- Chegg, Math Stack Exchange, or any other source where you can copy solutions is not allowed
- The homework needs to be typeset in “LaTeX” and uploaded through Gradescope in the iLearn Lecture page
- If a problem is similar to a problem in Hw #1 or Hw #2, you need to adapt the proof for this problem, not just refer or reproduce all the solution of the problem in the homework.

- **Problem 1:** Let $f, g \in R[a,b]$. Define

$$(f \vee g)(x) = \frac{(f+g)(x) + |(f-g)(x)|}{2}.$$

(Note that $(f \vee g)(x) = \max\{f(x), g(x)\}$.)

Show that $(f \vee g)$ is Riemann integrable in $[a,b]$.

- **Problem 2:** Let $g_n : [a,b] \to \mathbb{R}$, $g_n \geq 0$, and $g_n \in R[a,b]$ be a sequence of functions that satisfies

$$\lim_{n\to\infty} \int_a^b g_n(x)dx = 0.$$

a) Show that if $f \in R[a,b]$, then

$$\lim_{n\to\infty} \int_a^b f(x)g_n(x)dx = 0.$$

b) Show that if $f \in R[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0.$$

You can use that $\int x^n dx = \dfrac{x^{n+1}}{n+1}$.

- **Problem 3:**) Let $c > 0$. For a set $A \subseteq \mathbb{R}$, define cA by

$$cA = \{y \in \mathbb{R} \mid y = cx \text{ for some } x \in A\} \ .$$

a) Prove that $m^*(cA) = c\, m^*(A)$.

b) (Extra Credit) What happens in $\mathbb{R}^n$?

- **Problem 4:** If E_1, E_2 are Lebesgue measurable subets of $\mathbb{R}$, show that $E_1 \times E_2$ is Lebesgue measurable and

$$m(E_1 \times E_2) = m(E_1)m(E_2).$$