MATH 151C discussion, section 002 Ryan Ta
University of California, Riverside Spring 2021

Higher-Stakes Homework 1 solutions

1. Let $f, g \in R[a, b]$. Define

$$(f \vee g)(x) = \frac{(f+g)(x) + |(f-g)(x)|}{2}.$$

(Note that $(f \vee g)(x) = \max\{f(x), g(x)\}$.)

Show that $(f \vee g)$ is Riemann integrable in $[a, b]$.

Proof. We will use Theorem 0.3.1 of Nelson several times in this proof. Since we assume $f, g \in R([a, b])$, Theorem 0.3.1 of Nelson asserts that we have $f+g, f-g \in R([a, b])$. We also assume that an absolute value of a Riemann integrable function is again Riemann integrable; that said, we have $|f-g| \in R([a, b])$. By Theorem 0.3.1 of Nelson, we have $f+g+|f-g| \in R([a, b])$. By Theorem 0.3.1 of Nelson once last time, we conclude $f \vee g = \frac{f+g+|f-g|}{2} \in R([a, b])$, as desired. □

2. Let $g_n : [a, b] \to \mathbb{R}$, $g_n \geq 0$, and $g_n \in R[a, b]$ be a sequence of functions that satisfies

$$\lim_{n\to\infty} \int_a^b g_n(x)dx = 0.$$

(a) Show that if $f \in R[a, b]$, then

$$\lim_{n\to\infty} \int_a^b f(x)g_n(x)dx = 0.$$

Proof. Since we assume $f \in R([a, b])$, by definition of Riemann integrability, f is also bounded on $[a, b]$. So there exists a number $M > 0$ that satisfies $|f(x)| \leq M$ for all $x \in [a, b]$. Now let $\epsilon > 0$ be given. The definition of the hypothesis

$$\lim_{n\to\infty} \int_a^b g_n(x)\,dx = 0$$

is that there exists $N > 0$ such that, if $n \geq N$, then we have

$$\left|\int_a^b g_n(x)\,dx - 0\right| < \frac{\epsilon}{M}.$$

Also, since we assumed $g_n \geq 0$ in the hypothesis, we have $|g_n(x)| = g_n(x)$ for all $x \in [a, b]$. Now, using the triangle inequality for integrals, we have

$$\begin{aligned}
\left|\int_a^b f(x)g_n(x)\,dx - 0\right| &= \left|\int_a^b f(x)g_n(x)\,dx\right| \\
&\leq \int_a^b |f(x)||g_n(x)|\,dx \\
&\leq \int_a^b M|g_n(x)|\,dx \\
&= M\int_a^b g_n(x)\,dx \\
&= M\left|\int_a^b g_n(x)\,dx\right| \\
&= M\left|\int_a^b g_n(x)\,dx - 0\right| \\
&< M\frac{\epsilon}{M} \\
&= \epsilon,
\end{aligned}$$

and so we conclude

$$\lim_{n\to\infty} \int_a^b f(x)g_n(x)dx = 0,$$

as desired. □

(b) Show that if $f \in R[0, 1]$, then

$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0.$$

Hint: You can use $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1}$.

Proof. Since we assume $f \in R([0,1])$, its Riemann integral over $[0,1]$ exists and is

$$\begin{aligned}\int_0^1 x^n\,dx &= \left.\frac{x^{n+1}}{n+1}\right|_0^1 \\ &= \frac{(1)^{n+1}-(0)^{n+1}}{n+1} \\ &= \frac{1}{n+1},\end{aligned}$$

which implies

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 x^n\,dx &= \lim_{n\to\infty}\frac{1}{n+1} \\ &= 0.\end{aligned}$$

By setting $g_n(x) := x^n, a := 0, b := 1$, we can use part (a) to conclude

$$\lim_{n\to\infty}\int_0^1 x^n f(x)dx = 0,$$

as desired. $\square$

3. Let $c > 0$. For a set $A \subseteq \mathbb{R}$, define cA by

$$cA = \{y \in \mathbb{R} \mid y = cx \text{ for some } x \in A\}.$$

(a) Prove that $m^*(cA) = cm^*(A)$.

Proof. Consider the intervals $I_k \subseteq \mathbb{R}$ that satisfy $A \subseteq \bigcup_k I_k$. Then $S := \{I_k\}$ is a covering of A by closed intervals. Also, if we define

$$cI_k = \{y \in I_k \mid y = cx \text{ for some } x \in I_k\},$$

then $S_c := \{cI_k\}$ is a covering of cA by closed intervals. Now, for intervals we have

$$\begin{aligned}v(cI_k) &= cb_k - ca_k \\ &= c(b_k - a_k) \\ &= cv(I_k),\end{aligned}$$

which implies

$$\begin{aligned}\sigma(cS) &= \sum_k v(cI_k) \\ &= \sum_k cv(I_k) \\ &= c\sum_k v(I_k) \\ &= c\sigma(S).\end{aligned}$$

So we have

$$\begin{aligned}m^*(cA) &= \inf\{\sigma(cS) : cS \text{ is a covering of } cA \text{ by closed intervals}\} \\ &\leq \sigma(cS) \\ &= c\sigma(S),\end{aligned}$$

from which we take infimum over S both sides to obtain $m^*(A) \leq cm^*(A)$. Analogously, if $c \neq 0$ (the separate case $c = 0$ is trivial), then we have

$$\begin{aligned}m^*(A) &= \inf\{\sigma(S) : S \text{ is a covering of } A \text{ by closed intervals}\} \\ &\leq \sigma(S) \\ &= \frac{1}{c}c\sigma(S),\end{aligned}$$

from which we take infimum over S both sides to obtain $m^*(A) \leq \frac{1}{c}m^*(cA)$, or equivalently $m^*(cA) \geq cm^*(A)$. Combine the inequalities to conclude $m^*(cA) = cm^*(cA)$. $\square$

(b) (Extra Credit) What happens in $\mathbb{R}^n$?

Proof. Consider the intervals $I_k \subseteq \mathbb{R}^n$ that satisfy $A \subseteq \bigcup_k I_k$. Then $S := \{I_k\}$ is a covering of A by closed intervals. Also, if we define

$$cI_k = \{y \in I_k \mid y = cx \text{ for some } x \in I_k\},$$

then $S_c := \{cI_k\}$ is a covering of cA by closed intervals. Now, for intervals we have

$$\begin{aligned} v(cI_k) &= \prod_{i=1}^{n}(cb_i - ca_i) \\ &= \prod_{i=1}^{n}(c(b_i - a_i)) \\ &= c^n \prod_{i=1}^{n}(b_i - a_i) \\ &= c^n v(I_k), \end{aligned}$$

which implies

$$\begin{aligned} \sigma(cS) &= \sum_k v(cI_k) \\ &= \sum_k c^n v(I_k) \\ &= c^n \sum_k v(I_k) \\ &= c^n \sigma(S). \end{aligned}$$

So we have

$$\begin{aligned} m^*(cA) &= \inf\{\sigma(cS) : cS \text{ is a covering of } cA \text{ by closed intervals}\} \\ &\leq \sigma(cS) \\ &= c^n \sigma(S), \end{aligned}$$

from which we take infimum over S both sides to obtain $m^*(A) \leq cm^*(A)$. Analogously, if $c \neq 0$ (the separate case $c = 0$ is trivial), then we have

$$\begin{aligned} m^*(A) &= \inf\{\sigma(S) : S \text{ is a covering of } A \text{ by closed intervals}\} \\ &\leq \sigma(S) \\ &= \frac{1}{c^n} c^n \sigma(S), \end{aligned}$$

from which we take infimum over S both sides to obtain $m^*(A) \leq \frac{1}{c^n} m^*(cA)$, or equivalently $m^*(cA) \geq c^n m^*(A)$. Combine the inequalities to conclude $m^*(cA) = c^n m^*(cA)$. □

4. If E_1, E_2 are Lebesgue measurable subets of $\mathbb{R}$, show that $E_1 \times E_2$ is Lebesgue measurable and

$$m(E_1 \times E_2) = m(E_1)m(E_2).$$

Proof. Since $E_1, E_2 \subseteq \mathbb{R}$ are Lebesgue measurable, there exist open sets $G_1, G_2 \subseteq \mathbb{R}$ that satisfy $m^*(G_1 \setminus E_1) < \epsilon$ and $m^*(G_2 \setminus E_2) < \epsilon$. Now consider the open set $G_1 \times G_2 \subseteq \mathbb{R}^2$. Notice that we can write

$$(G_1 \times G_2) \setminus (E_1 \times E_2) = ((G_1 \setminus E_1) \times E_2) \cup (E_1 \times (G_2 \setminus E_2)) \cup ((G_1 \setminus E_1) \times (G_2 \setminus E_2)),$$

Also, for any two sets $A, B \subseteq \mathbb{R}$, let $S = \{I_k\}$ be a covering of A by closed intervals I_k, and let $T = \{J_\ell\}$ be a covering of B by closed intervals J_ℓ. Then we have

$$\begin{aligned} m^*(A \times B) &= \inf\{\sigma(S \times T) : S \times T \text{ is a covering of } A \times B \text{ by closed intervals}\} \\ &\leq \sigma(S \times T) \\ &= \sum_k \sum_\ell v(I_k \times J_\ell) \\ &= \sum_k \sum_\ell v(I_k)v(J_\ell) \\ &= \sum_k v(I_k) \sum_\ell v(J_\ell) \\ &= \sigma(S)\sigma(T), \end{aligned}$$

from which we take the infimum over both sides to conclude

$$\begin{aligned}
m^*(A \times B) &= \inf\{m^*(A \times B) : S, T \text{ are coverings of } A, B \text{ by closed intervals, respectively}\} \\
&\leq \inf\{\sigma(S)\sigma(T) : S, T \text{ are coverings of } A, B \text{ by closed intervals, respectively}\} \\
&= m^*(A)m^*(B).
\end{aligned}$$

Likewise, we have

$$\begin{aligned}
m^*(A)m^*(B) &= \inf\{\sigma(S) : S \text{ is a covering of } A \text{ by closed intervals}\} \\
&\qquad \times \inf\{\sigma(T) : T \text{ is a covering of } B \text{ by closed intervals}\} \\
&\leq \sigma(S)\sigma(T) \\
&= \sum_k v(I_k) \sum_\ell v(J_\ell) \\
&= \sum_k \sum_\ell v(I_k)v(J_\ell) \\
&= \sum_k \sum_\ell v(I_k \times J_\ell) \\
&= \sigma(S \times T),
\end{aligned}$$

from which we take the infimum over both sides to conclude

$$\begin{aligned}
m^*(A)m^*(B) &= \inf\{m^*(A)m^*(B) : S, T \text{ are coverings of } A, B \text{ by closed intervals, respectively}\} \\
&\leq \inf\{\sigma(S \times T) : S, T \text{ are coverings of } A, B \text{ by closed intervals, respectively}\} \\
&= m^*(A \times B).
\end{aligned}$$

Combine $m^*(A \times B) \leq m^*(A)m^*(B)$ and $m^*(A \times B) \geq m^*(A)m^*(B)$ to conclude $m^*(A \times B) = m^*(A)m^*(B)$. So we have

$$\begin{aligned}
m^*((G_1 \times G_2) \setminus (E_1 \times E_2)) &= m^*((G_1 \setminus E_1) \times E_2) \cup (E_1 \times (G_2 \setminus E_2)) \cup ((G_1 \setminus E_1) \times (G_2 \setminus E_2)) \\
&\leq m^*((G_1 \setminus E_1) \times E_2) + m^*(E_1 \times (G_2 \setminus E_2)) + m^*((G_1 \setminus E_1) \times (G_2 \setminus E_2)) \\
&= m^*(G_1 \setminus E_1)m^*(E_2) + m^*(E_1)m^*(G_2 \setminus E_2) + m^*(G_1 \setminus E_1)m^*(G_2 \setminus E_2) \\
&< \epsilon m^*(E_2) + m^*(E_1)\epsilon + \epsilon\epsilon \\
&= (m^*(E_1) + m^*(E_2) + \epsilon)\epsilon.
\end{aligned}$$

Also, notice that $G_1 \times G_2$ is open because G_1, G_2 are open. Therefore, $E_1 \times E_2$ is Lebesgue measurable, and we readily have

$$\begin{aligned}
m(E_1 \times E_2) &= m^*(E_1 \times E_2) \\
&= m^*(E_1)m^*(E_2) \\
&= m(E_1)m(E_2),
\end{aligned}$$

as desired. $\square$