MATH 151C discussion, section 002 Ryan Ta
University of California, Riverside Spring 2021

Homework 1 solutions

1. (Chapter 0, Problem 3) Let $f, g \in R[a, b]$ with $f(x) \le g(x)$ for all $x \in [a, b]$. Prove that
$$\int_a^b f(x)dx \le \int_a^b g(x)dx.$$

Proof. Consider the partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$. Since we have $f(x) \le g(x)$ for all $x \in [a, b]$, we have in particular $f(x) \le g(x)$ for any $x \in [x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$. Take the infimum and supremum of f and g over $[x_{i-1}, x_i]$ to conclude
$$\begin{aligned} m_i^f &:= \inf_{x \in [x_{i-1}, x_i]} f(x) \\ &\le \inf_{x \in [x_{i-1}, x_i]} g(x) \\ &=: m_i^g, \end{aligned}$$

which implies
$$\begin{aligned} L(f, P) &= \sum_{i=0}^{n} m_i^f (x_i - x_{i-1}) \\ &\le \sum_{i=0}^{n} m_i^g (x_i - x_{i-1}) \\ &= L(g, P). \end{aligned}$$

But we also assume $f, g \in R[a, b]$, which means
$$\begin{aligned} \int_a^b f(x)\, dx &= L(f, P) \\ &\le L(g, P) \\ &= \int_a^b g(x)\, dx \end{aligned}$$

as desired. (Observe that, because we assume $f, g \in R([a, b])$, we have $L(f, P) = U(f, P)$ annd $L(g, P) = U(g, P)$; the above argument using $U(f, P) \le U(g, P)$ is exactly the same as the one using $L(f, P) \le L(g, P)$.) □

2. (Chapter 0, Problem 5) Assume $f \in R[a, b]$.

(a) Let $c \in [a, b]$. Suppose g is defined on $[a, b]$ and $g(x) = f(x)$ for all $x \neq c$. Show $g \in R[a, b]$.

Proof. First, we write $g = (g - f) + f$. By Theorem 0.3.1 of Nelson, we only need to show $g - f \in R([a, b])$. Let $\epsilon > 0$ be given (and sufficiently small). If $c \in [a, b]$ is an interior point, consider the partition $P := \{a, c - \epsilon, c + \epsilon, b\}$. Then we have
$$\begin{aligned} U(g - f, P) &= 0((c - \epsilon) - a) + \max\{g - f, 0\}((c + \epsilon) - (c - \epsilon)) + 0(b - (c + \epsilon)) \\ &= 2\epsilon \max\{g - f, 0\} \end{aligned}$$

and
$$\begin{aligned} L(g - f, P) &= 0((c - \epsilon) - a) + \min\{g - f, 0\}((c + \epsilon) - (c - \epsilon)) + 0(b - (c + \epsilon)) \\ &= 2\epsilon \min\{g - f, 0\}, \end{aligned}$$

which implies the difference
$$\begin{aligned} U(g - f, P) - L(g - f, P) &= 2\epsilon \max\{g - f, 0\} - 2\epsilon \min\{g - f, 0\} \\ &= 2(\max\{g - f, 0\} - \min\{g - f, 0\})\epsilon, \end{aligned}$$

and so we conclude $g - f \in R([a, b])$. □

(b) Suppose g differs from f at a finite number of points. Show $g \in R[a, b]$.

Proof. We will prove this by induction. We have already established the base case in part (a). For the induction step, assume that, if g differs from f at n points $r_1, r_2, \ldots, r_n$, then $g \in R([a, b])$. Consider a new function h on $[a, b]$. So we will show that, if h differs from f at $n + 1$ points $r_1, r_2, \ldots, r_{n+1}$, then $h \in R([a, b])$. From the construction of h, we see that h differs from g at r_{n+1}. Apply the argument of part (a) to h and the point $c = r_{n+1}$ to complete the proof by induction. □

(c) Does this extend to the case where g and f differ at a countable number of points? Prove or give a counterexample.

Proof. No. Let $f = 0$ and let g be the Dirichlet function

$$g(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [a, b], \\ 0 & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [a, b]. \end{cases}$$

Since $\mathbb{Q}$ is a countable set, it follows that g differs from f at a countable number of points. However, we have $f \in R([a, b])$ and $g \notin R([a, b])$. The proofs of these statements can be found in lecture and in the Nelson textbook, so I will not include their proofs again here. □

3. (Chapter 0, Problem 6) Let $\{f_n\}$ be a sequence of functions with $f_n \in R[a, b]$ for each n. Suppose the sequence $\{f_n\}$ converges uniformly to f on $[a, b]$. Show that $f \in R[a, b]$.

Proof. Let $\epsilon > 0$ be given. Since $\{f_n\}$ converges uniformly to f on $[a, b]$, there exists $N > 0$ such that, if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in [a, b]$, which implies in particular $|M_i^f - M_i^{f_n}| \leq \epsilon$ and $|m_i^f - m_i^{f_n}| \leq \epsilon$, and so we have

$$\begin{aligned} |U(f, P) - U(f_n, P)| &\leq \sum_{i=1}^{k} |M_i^{f_n} - M_i^f|(x_k - x_{k-1}) \\ &\leq \sum_{i=1}^{k} \epsilon(x_k - x_{k-1}) \\ &= \epsilon(b - a) \end{aligned}$$

and

$$\begin{aligned} |L(f, P) - L(f_n, P)| &\leq \sum_{i=1}^{k} |m_i^{f_n} - m_i^f|(x_k - x_{k-1}) \\ &\leq \sum_{i=1}^{k} \epsilon(x_k - x_{k-1}) \\ &= \epsilon(b - a) \end{aligned}$$

Since we have $f_n \in R([a, b])$, there exists a partition P that satisfies $U(f_n, P) - L(f_n, P) < \epsilon$. So we have

$$\begin{aligned} U(f, P) - L(f, P) &= U(f, P) - U(f_n, P) + U(f_n, P) - L(f_n, P) + L(f_n, P) - L(f, P) \\ &\leq |U(f, P) - U(f_n, P)| + U(f_n, P) - L(f_n, P) + |L(f_n, P) - L(f, P)| \\ &< \epsilon(b - a) + \epsilon + \epsilon(b - a) \\ &= (2(b - a) + 1)\epsilon. \end{aligned}$$

So, by Theorem 0.2.4 of Nelson, we conclude $f \in R([a, b])$. □

4. (Chapter 0, Problem 7) Prove or modify and then prove: Let $f \in B[a, b]$. Define

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f^-(x) = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ -f(x) & \text{otherwise} \end{cases}$$

Then $f \in R[a, b]$ if and only if both $f^+ \in R[a, b]$ and $f^- \in R[a, b]$.

Proof. Suppose $f^+, f^- \in R([a, b])$. Observe based on the definitions of f^+, f^- that, for any $f \in B([a, b])$, we have

$$f(x) = f^+(x) - f^-(x)$$

for all $x \in [a, b]$. By Theorem 0.3.1 of Nelson, we conclude $f = f^+ - f^- \in R([a, b])$.

Conversely, suppose $f \in R([a, b])$. Then for all $\epsilon > 0$, there exists a partition P of $[a, b]$ that satisfies $U(f, P) - L(f, P) < \epsilon$. First, we notice

$$|f(x)| = f^+(x) + f^-(x).$$

So we have

$$\begin{aligned} f^+(x) &= \frac{|f(x)| + f(x)}{2}, \\ f^-(x) &= \frac{|f(x)| - f(x)}{2}. \end{aligned}$$

By Theorem 0.3.1 of Nelson, we have $f^+, f^- \in R([a, b])$, provided that we prove $|f| \in R([a, b])$. (Check the Lebl textbook; maybe there is a proof for $|f| \in R([a, b])$; I am not including that proof here.) □

5. (Chapter 0, Problem 11) Let $\{r_1, r_2, \ldots, r_n, \ldots\}$ be a counting of the rational numbers in the interval $[0, 1]$. For each natural number k, define the function f_k by

$$f_k(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \ldots, r_k\} \\ 0 & \text{otherwise} \end{cases}$$

(a) Find f, the pointwise limit of the sequence $\{f_k\}$.

Proof. The pointwise limit of $\{f_k\}$ is f, given by

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1], \\ 0 & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1], \end{cases}$$

which is the Dirichlet function on $[0, 1]$. □

(b) Show that $f_k \in R[0, 1]$ for each k.

Proof. Consider g, defined by $g(x) := 0$ for all $x \in [0, 1]$. Then we have $g \in R([0, 1])$ because, for any $\epsilon > 0$, we have $U(f, P) - L(f, P) = 0 - 0 < \epsilon$. Also, g differs from f at only a finite number of points, namely at $x = r_1, r_2, \ldots, r_k$. By Exercise 2(b) (Chapter 0, Problem 5(b) of Nelson), we conclude $f_k \in R([0, 1])$ for each k. □

(c) In general, if $\{f_n\}$ is a sequence of Riemann integrable functions which converge pointwise to f, is f Riemann integrable?

Proof. No. For instance, if f is the Dirichlet function as mentioned in part (a), then we have $f \notin R([0, 1])$. I proved this already in my solution to Exercise 2(c). □