Homework 2 solutions

1. (Chapter 1, Problem 2) Let A be a countable set of real numbers. Use the definition of outer measure to show $m^*(A) = 0$.

   Proof. We can assume that A is countably infinite, because the argument for the case of A being finite is similar. Since A is countable, there exists a one-to-one correspondence of each element in A with the natural numbers. So we can let $\{x_k\}_{k=1}^{\infty}$ be a sequence of points $x_k \in A$ that enumerate the countable set A. Given $\epsilon > 0$, consider the closed interval $I_k := [x_k - \frac{\epsilon}{2^{k+2}}, x_k + \frac{\epsilon}{2^{k+2}}]$, and consider the collection of closed intervals $S = \{I_k\}$. Then we have

$$\begin{aligned}\sigma(S) &= \sum_{k=1}^{\infty} v(I_k) \\ &= \sum_{k=1}^{\infty} \left(\left(x_k + \frac{\epsilon}{2^{k+1}}\right) - \left(x_k - \frac{\epsilon}{2^{k+2}}\right)\right) \\ &= \sum_{k=1}^{\infty} \frac{2\epsilon}{2^{k+2}} \\ &= \frac{\epsilon}{2} \sum_{k=1}^{\infty} \frac{1}{2^k} \\ &= \frac{\epsilon}{2} \cdot 1 \\ &= \frac{\epsilon}{2} \\ &< \epsilon,\end{aligned}$$

   which implies the upper bound of the Lebesgue outer measure

$$\begin{aligned}m^*(A) &= \inf\{\sigma(S) : S \text{ is a covering of } A \text{ by closed intervals}\} \\ &\leq \sigma(S) \\ &< \epsilon.\end{aligned}$$

   Finally, since $\epsilon > 0$ is arbitrary, we conclude $m^*(A) = 0$. □

2. (Chapter 1, Problem 3) Let S and T be coverings of a set A by intervals.

   (a) Explain why $S \cup T$ is also a covering of A by intervals.

   Proof. Since S, T are coverings of A by intervals, we can write $S = \{I_k\}$ and $T = \{J_k\}$, where we have $A \subseteq \bigcup_k I_k$ and $A \subseteq \bigcup_k J_k$. Then we have $A \subseteq (\bigcup_k I_k) \cup (\bigcup_k J_k) = \bigcup_k (I_k \cup J_k)$. In other words, if we write $S \cup T = \{\bigcup_k (I_k \cup J_k)\}$, then $S \cup T$ is a covering of A by intervals. □

   (b) Show that $\sigma(S \cup T) \leq \sigma(S) + \sigma(T)$.

   Proof. If intervals are closed: We have

$$\begin{aligned}\sigma(S \cup T) &= \sum_{k=1}^{\infty} v(I_k \cup J_k) \\ &= \sum_{k=1}^{\infty} m^*(I_k \cup J_k) \\ &\leq \sum_{k=1}^{\infty} (m^*(I_k) + m^*(J_k)) \\ &= \sum_{k=1}^{\infty} (v(I_k) + v(J_k)) \\ &= \sum_{k=1}^{\infty} v(I_k) + \sum_{k=1}^{\infty} v(J_k) \\ &= \sigma(S) + \sigma(T)\end{aligned}$$

□

3. (Chapter 1, Problem 4) Show that for $c \in \mathbb{R}$ and fixed k, the set (known as a hyperplane in $\mathbb{R}^n$)

$$A = \{x = (x_1, x_2, \ldots, x_k, \ldots, x_n) \in \mathbb{R}^n \mid x_k = c\}$$

has Lebesgue outer measure 0.

Proof. Let $\epsilon > 0$ be given. For all $l = 1, 2, 3, \ldots$ and some fixed $k = 1, \ldots, n$, consider the closed interval $I_l \in \mathbb{R}^n$ defined by

$$I_l := \{x = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-1} : -l \leq x_i \leq l, i = 1, \ldots, k-1, k+1, \ldots, n\} \times \left[c - \frac{\epsilon}{2^{l+2}(2l)^{n-1}}, c + \frac{\epsilon}{2^{l+2}(2l)^{n-1}}\right].$$

and consider the set $S = \{I_l\}_{l=1}^{\infty}$. Then we have

$$\begin{aligned}
\sigma(S) &= \sum_{l=1}^{\infty} v(I_l) \\
&= \sum_{l=1}^{\infty} \left(\prod_{k=1}^{n-1} (l - (-l))\right) \left(c + \frac{\epsilon}{2^{l+2}(2l)^{n-1}} - \left(c - \frac{\epsilon}{2^{l+2}(2l)^{n-1}}\right)\right) \\
&= \sum_{l=1}^{\infty} \left(\prod_{k=1}^{n-1} (2l)\right) \frac{2\epsilon}{2^{l+2}(2l)^{n-1}} \\
&= \sum_{l=1}^{\infty} (2l)^{n-1} \frac{\epsilon}{2^{l+1}(2l)^{n-1}} \\
&= \frac{\epsilon}{2} \sum_{k=1}^{\infty} \frac{1}{2^l} \\
&= \frac{\epsilon}{2} \cdot 1 \\
&< \epsilon,
\end{aligned}$$

which implies the upper bound of the Lebesgue outer measure

$$\begin{aligned}
m^*(A) &= \inf\{\sigma(S) : S \text{ is a covering of } A \text{ by closed intervals}\} \\
&\leq \sigma(S) \\
&< \epsilon.
\end{aligned}$$

Finally, since $\epsilon > 0$ is arbitrary, we conclude $m^*(A) = 0$. □

4. (Chapter 1, Problem 5) Suppose A and B are both Lebesgue measurable. Prove that if both A and B have measure zero, then $A \cup B$ is Lebesgue measurable and $m(A \cup B) = 0$.

   (a) Do this directly from the definition.

   Proof. We have

$$\begin{aligned}
0 &\leq m^*(A \cup B) \\
&\leq m^*(A) + m^*(B) \\
&= 0 + 0 \\
&= 0,
\end{aligned}$$

   which implies $m^*(A \cup B) = 0$. So $A \cup B$ is Lebesgue measurable and satisfies $m(A \cup B) = 0$.
   By subadditivity of the Lebesgue outer measure, we have

$$\begin{aligned}
0 &\leq m(A \cup B) \\
&= m^*(A \cup B) \\
&\leq m^*(A) + m^*(B) \\
&= m(A) + m(B) \\
&= 0 + 0 \\
&= 0,
\end{aligned}$$

   which implies $m(A \cup B) = 0$. □

   (b) Give a shorter proof by using Theorem 1.2.5.

Proof. By Theorem 1.2.5, $A \cup B$ we have

$$\begin{aligned} 0 &\le m(A \cup B) \\ &\le m(A) + m(B) \\ &= 0 + 0 \\ &= 0, \end{aligned}$$

which implies $m(A \cup B) = 0$. □

5. (Chapter 1, Problem 6) Suppose A has Lebesgue measure zero and $B \subseteq A$. Prove B is Lebesgue measurable and $m(B) = 0$.

Proof. Since A has Lebesgue measure zero, there exists an open set G so that $A \subseteq G$ and $m^*(G \setminus A) < \epsilon$. So we have $B \subseteq A \subseteq G$ and $G \setminus B = (G \setminus A) \cup (A \setminus B)$ (disjoint union). Using the subadditivity of the Lebesuge outer measure, we obtain

$$\begin{aligned} m^*(G \setminus B) &= m^*((G \setminus A) \cup (A \setminus B)) \\ &\le m^*(G \setminus A) + m^*(A \setminus B) \\ &< \epsilon + m^*(A \setminus B) \\ &\le \epsilon + m^*(A) \\ &= \epsilon + m(A) \\ &= \epsilon + 0 \\ &= \epsilon, \end{aligned}$$

which means B is Lebesgue measurable. Also, we have

$$\begin{aligned} 0 &\le m(B) \\ &= m^*(B) \\ &\le m^*(A) \\ &= m(A) \\ &= 0, \end{aligned}$$

which implies $m(B) = 0$. □

6. (Chapter 1, Problem 11) Let A be a subset of $\mathbb{R}$ and $c \in \mathbb{R}$. Define $A + c$ to be the set

$$A + c = \{x + c \mid x \in A\}$$

(a) Prove $m^*(A + c) = m^*(A)$.

Proof. Let S and S_c be respective coverings of A and of $A + c$ by closed intervals. Then we can write $S := \{I_k\}$, where I_k are closed intervals with $A \subseteq \bigcup_k I_k$. If we write $S_c := \{I_k + c\}$, then $I_k + c$ are also closed intervals with $A + c \subseteq \bigcup_k (I_k + c)$, and so S_c is a covering of $A + c$ by closed intervals. So we have

$$\begin{aligned} \sigma(S) &= \sum_k v(I_k) \\ &= \sum_k v(I_k + c) \\ &= \sigma(S_c). \end{aligned}$$

As a result, we obtain

$$\begin{aligned} m^*(A) &= \inf\{\sigma(S) : S \text{ is a covering of } A \text{ by closed intervals}\} \\ &\le \sigma(S) \\ &= \sigma(S_c), \end{aligned}$$

from which we can take infimums over S_c to obtain

$$\begin{aligned} m^*(A) &= \inf\{m^*(A) : S \text{ is a covering of } A + c \text{ by closed intervals}\} \\ &\le \inf\{\sigma(S_c) : S \text{ is a covering of } A + c \text{ by closed intervals}\} \\ &= m^*(A + c). \end{aligned}$$

Likewise, we obtain

$$\begin{aligned} m^*(A + c) &= \inf\{\sigma(S_c) : S_c \text{ is a covering of } A + c \text{ by closed intervals}\} \\ &\le \sigma(S_c) \\ &= \sigma(S), \end{aligned}$$

from which we can take infimums over S_c to obtain

$$\begin{aligned} m^*(A+c) &= \inf\{m^*(A+c) : S_c \text{ is a covering of } A+c \text{ by closed intervals}\} \\ &\leq \inf\{\sigma(S) : S \text{ is a covering of } A \text{ by closed intervals}\} \\ &= m^*(A). \end{aligned}$$

So we have $m^*(A) \leq m^*(A+c)$ and $m^*(A) \geq m^*(A+c)$, which together imply $m^*(A) = m^*(A+c)$. □

(b) Prove that $A+c$ is Lebesgue measurable if and only if A is Lebesgue measurable.

Proof. Suppose that $A+c$ is Lebesgue measurable. Then there exists an open set $G \subseteq \mathbb{R}$ so that $A+c \subseteq G$ and $m^*(G \setminus (A+c)) < \epsilon$. Define the set $G+c := \{x+c : x \in G\}$, which is also an open set in $\mathbb{R}$. Then we have

$$\begin{aligned} (G-c) \setminus A &= \{x-c : x-c \in (G-c) \setminus A\}, \\ &= \{x : x \in G \setminus (A+c)\} \\ &= G \setminus (A+c), \end{aligned}$$

which implies, along with part (a),

$$\begin{aligned} m^*((G-c) \setminus A) &= m^*(G \setminus (A+c)) \\ &< \epsilon. \end{aligned}$$

So A is Lebesgue measurable.

Conversely, suppose that A is Lebesgue measurable. Then there exists an open set $G \subseteq \mathbb{R}$ so that $A \subseteq G$ and $m^*(G \setminus A) < \epsilon$. Define the set $G+c := \{x+c : x \in G\}$, which is also an open set in $\mathbb{R}$. Then we have

$$\begin{aligned} (G+c) \setminus (A+c) &= \{x+c : x \in (G+c) \setminus (A+c)\} \\ &= \{x : x \in G \setminus A\} \\ &= G \setminus A, \end{aligned}$$

which implies, along with part (a),

$$\begin{aligned} m^*((G+c) \setminus (A+c)) &= m^*(G \setminus A \\ &< \epsilon. \end{aligned}$$

So $A+c$ is Lebesgue measurable. □