Homework 3 solutions

1. (Chapter 1, Problem 6) Let E be a measurable subset of $\mathbb{R}^n$. Show that given $\epsilon > 0$ there is a closed set F and an open set G with $F \subseteq E \subseteq G$ and $m(G \backslash F) < \epsilon$.

Proof. Let $\epsilon > 0$ be given. Since $E \subseteq \mathbb{R}^n$ is measurable, there exists an open set G with $E \subseteq G$ and $m(G \setminus E) < \frac{\epsilon}{2}$. Also, by Proposition 1.2.23 of Nelson, there exists a closed set $F \subseteq E$ with $m^*(E \setminus F) < \frac{\epsilon}{2}$. Since $F \subseteq E$ is closed, it follows that $E \setminus F \subseteq E$ is open. According to Example 1.2.23 of Nelson, which states that every open set is measurable; in particular, $E \setminus F$ is measurable. So we have $m(E \setminus F) = m^*(E \setminus F) < \frac{\epsilon}{2}$. Finally, when writing $G \setminus F = (G \setminus E) \cup (E \setminus F)$, where the union is disjoint, we have

$$\begin{aligned} m(G \setminus F) &= m((G \setminus E) \cup (E \setminus F)) \\ &= m(G \setminus E) + m(E \setminus F) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

as desired. □

2. (Chapter 1, Problem 24) Let A be a subset of $\mathbb{R}^n$. Show that there is a set H of type G_δ so that

$$A \subseteq H \text{ and } m^*(A) = m^*(H) .$$

Proof. By Theorem 1.1.13 of Nelson, for every $\epsilon > 0$ there exists an open set G such that $A \subseteq G$ and

$$m^*(G) < m^*(A) + \epsilon.$$

Now choose in particular $\epsilon := \frac{1}{n}$ for any positive integer n. Then there exist countably many open sets H_n satisfying $A \subseteq H_n$ and

$$m^*(H_n) < m^*(A) + \frac{1}{n}$$

Now define $H := \bigcap_{n=1}^{\infty} H_n$. Then H is the intersection of a countable collection of open sets H_n, and so according to Definition 1.2.20 of Nelson H is of type G_δ. Also, $A \subseteq H_n$ for all positive integers n implies $A_n \subseteq H$, as desired. Furthermore, $A \subseteq H$ implies $m^*(A) \leq m^*(H)$ and $H = \bigcap_{n=1}^{\infty} H_n \subseteq H_n$ implies $m^*(H) \leq m^*(H_n)$, both of which is justified by Proposition 1.1.8 of Nelson. So we have

$$\begin{aligned} m^*(A) &\leq m^*(H) \\ &\leq m^*(H_n) \\ &< m^*(A) + \frac{1}{n}, \end{aligned}$$

which holds for all positive integers n, and so we conclude $m^*(A) \leq m^*(H) \leq m^*(A)$, which implies $m^*(A) = m^*(H)$, as desired. □

3. (Chapter 2, Problem 1) Let $E \subseteq [a, b]$ and let $\mathcal{X}_E$ be the characteristic function of E. Prove that $\mathcal{X}_E(x)$ is a measurable function if and only if E is a measurable set.

Proof. Suppose χ_E is a measurable function. Then, given $s \in \mathbb{R}$, the set $\{x \in [a, b] : \chi_E(x) > s\}$ is measurable. In particular, if we choose any $s \leq 0$, then we would have $\{x \in [a, b] : \chi_E(x) > s\} = E$, and so we conclude that E is measurable.

Conversely, suppose E is a measurable set.

(i) If $s \geq 1$, then $\{x \in [a, b] : \chi_E(x) > s\} = \varnothing$, which is a Lebesgue measurable set.

(ii) If $0 \leq s < 1$, then $\{x \in [a, b] : \chi_E(x) > s\} = E$, which is, by our assumption, a Lebesgue measurable set.

(ii) If $s < 0$, then $\{x \in [a, b] : \chi_E(x) > s\} = [a, b]$, which is a Lebesgue measurable set.

So χ_E is a measurable function. □

4. (Chapter 2, Problem 3) Let $[c, d] \subseteq [a, b]$. Show that if f is measurable on $[a, b]$, then f is measurable on $[c, d]$.

Proof. Since f is measurable on $[a,b]$, the set $E_{a,b} = \{x \in [a,b] : f(x) > s\}$ is measurable. Notice that we have

$$\begin{aligned} E_{c,d} &= \{x \in [c,d] : f(x) > s\} \\ &= \{x \in [a,b] : f(x) > s\} \cap [c,d] \\ &= E_{a,b} \cap [c,d] \end{aligned}$$

Recall that $E_{a,b}$ is measurable by assumption. Also, $[c,d]$ is measurable because the open interval $G := (c - \frac{\epsilon}{4}, d + \frac{\epsilon}{4}) \subseteq \mathbb{R}$ is an open set satisfying, for any $\epsilon > 0$,

$$\begin{aligned} m^*(G \setminus [c,d]) &= m^*\left(\left(c - \frac{\epsilon}{4}, d + \frac{\epsilon}{4}\right) \setminus [c,d]\right) \\ &= m^*\left(\left(c - \frac{\epsilon}{4}, c\right)\right) \cup \left(\left(d, d + \frac{\epsilon}{4}\right)\right) \\ &\leq m^*\left(\left(c - \frac{\epsilon}{4}, c\right)\right) + m^*\left(\left(d, d + \frac{\epsilon}{4}\right)\right) \\ &= \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &= \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

By Proposition 1.2.19 of Nelson, which asserts that any intersection of measurable sets is again measurable, we conclude that $E_{c,d}$ is measurable. □

5. (Chapter 2, Problem 4) Find an example of a pointwise bounded sequence of measurable functions $\{f_n\}$ on $[0,1]$ such that each $f_n(x)$ is a bounded function but $f^*(x) = \limsup_{n\to\infty} f_n(x)$ is not a bounded function.

   Proof. Define for instance

   $$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x < \frac{1}{n}, \\ \frac{1}{x} & \text{if } \frac{1}{n} \leq x \leq 1. \end{cases}$$

   Then, for any $s \in \mathbb{R}$, the set $\{x \in [0,1] : f_n(x) > s\}$ is one of $[0,1]$, an subinterval of $[0,1]$, or the empty set, all of which are measurable. So f_n is a measurable function. It is also bounded because we have $|f_n(x)| \leq n$. But we have

   $$\begin{aligned} f^*(x) &= \limsup_{n\to\infty} f_n(x) \\ &= \lim_{n\to\infty} n \\ &= \infty, \end{aligned}$$

   meaning that f^* is not a bounded function. □

6. (Chapter 2, Problem 8) Suppose f is measurable on $I = [a,b]$ and $f(x) \geq 0$ a.e. on I. Prove that if the set $\{x \in I \mid f(x) > 0\}$ has positive measure, then for some positive integer n the set

   $$E_n = \left\{x \in I \mid f(x) > \frac{1}{n}\right\}$$

   has positive measure.

   Proof. Suppose to the contrary that E_n does not have positive measure; in other words, suppose $m(E_n) = 0$. Since f is measurable, for any $s \in \mathbb{R}$, the set $\{x \in I : f(x) > s\}$ is a measurable set. In particular, $\{x \in I : f(x) > 0\}$ and $\{x \in I : f(x) > \frac{1}{n}\}$ for any positive integer n are measurable sets. Furthermore, notice that we have

   $$\{x \in I : f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n.$$

   So we have

   $$\begin{aligned} m(\{x \in I : f(x) > 0\}) &= m\left(\bigcup_{n=1}^{\infty} E_n\right) \\ &\leq \sum_{n=1}^{\infty} m(E_n) \\ &= \sum_{n=1}^{\infty} 0 \\ &= 0, \end{aligned}$$

   meaning that the set $\{x \in I : f(x) > 0\}$ has nonpositive measure. But this contradicts our assumption that the set $\{x \in I : f(x) > 0\}$ has positive measure. So we conclude that E_n has positive measure. □