Homework 4 solutions

1. (Chapter 2, Problem 13) Let f and g be bounded, Lebesgue integrable functions on $[a, b]$. Show that $f + g$ is Lebesgue integrable on $[a, b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Hint: Exercise 11 might be useful.

Proof. Let $\epsilon > 0$ be given. Since f, g are Lebesgue integrable functions on $[a, b]$, we have

$$\overline{\int_a^b} f = \int_a^b f = \underline{\int_a^b} f,$$
$$\overline{\int_a^b} g = \int_a^b g = \underline{\int_a^b} g$$

and, by Lemma 2.2.11 of Nelson, there exists a measurable partition P of $[a, b]$ that satisfies

$$U(f,P) - L(f,P) < \frac{\epsilon}{2},$$
$$U(g,P) - L(g,P) < \frac{\epsilon}{2}.$$

By Exercise 11 of Nelson, we have

$$U(f+g,P) \le U(f,P) + U(g,P),$$
$$L(f+g,P) \ge L(f,P) + L(g,P)$$

So we have

$$\begin{aligned} U(f+g,P) - L(f+g,P) &\le (U(f,P) + U(g,P)) - (L(f,P) + L(g,P)) \\ &= (U(f,P) - L(f,P)) + (U(g,P) - L(g,P)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

and so by Lemma 2.2.11 of Nelson we conclude that $f + g$ is Lebesgue integrable on $[a, b]$. We also have

$$\begin{aligned} \overline{\int_a^b} (f+g) &\le U(f+g,P) \\ &= U(f,P) + U(g,P), \end{aligned}$$

from which we take supremums over P is

$$\begin{aligned} \overline{\int_a^b} (f+g) &\le \overline{\int_a^b} f + \overline{\int_a^b} g \\ &= \int_a^b f + \int_a^b g \end{aligned}$$

and

$$\begin{aligned} \underline{\int_a^b} (f+g) &\ge L(f+g,P) \\ &= L(f,P) + L(g,P). \end{aligned}$$

We also have

$$\begin{aligned} \int_a^b f + \int_a^b g &\le \underline{\int_a^b} (f+g) \\ &\le \overline{\int_a^b} (f+g), \end{aligned}$$

from which we take supremums over P is

$$\begin{aligned}\underline{\int_a^b}(f+g) &\leq \underline{\int_a^b} f + \underline{\int_a^b} g \\ &= \int_a^b f + \int_a^b g.\end{aligned}$$

Combine the results together to conclude

$$\begin{aligned}\int_a^b (f+g) &= \underline{\int_a^b}(f+g) \\ &= \overline{\int_a^b}(f+g) \\ &= \int_a^b f + \int_a^b g,\end{aligned}$$

as desired. □

2. (Chapter 2, Problem 14) Let h be a bounded function that is zero a.e. in $[a,b]$. Show that h is Lebesgue integrable on $[a,b]$ and

$$\int_a^b h = 0.$$

Proof. Since, for any $s \in \mathbb{R}$, we have

$$\{x \in [a,b] : f(x) > s\} = \begin{cases}[a,b] & \text{if } s \geq 0, \\ \varnothing & \text{if } s < 0,\end{cases}$$

and $[a,b], \varnothing$ are both Lebesgue measurable sets, it follows that the zero function is a Lebesgue measurable function on $[a,b]$. Now, since we assume $h = 0$ a.e. in $[a,b]$, Proposition 2.1.9 of Nelson asserts that h is also Lebesgue measurable on $[a,b]$. Furthermore, by Proposition 2.2.12 of Nelson, h is also Lebesuge integrable on $[a,b]$. Now, $h = 0$ a.e. in $[a,b]$ means that the set $Z := \{x \in [a,b] : h(x) \neq 0\}$ has Lebesgue measure zero; that is, $m(Z) = 0$. Therefore, assuming Exercise 19, we have

$$\begin{aligned}\int_a^b h &= \int_Z h + \int_{[a,b]\setminus Z} h \\ &= \int_Z h + \int_{[a,b]\setminus Z} 0 \\ &= \int_Z h,\end{aligned}$$

which implies

$$\begin{aligned}\left|\int_a^b h\right| &\leq \int_Z |h| \\ &\leq \int_Z \sup_Z |h| \\ &= \sup_Z |h| \int_Z 1 \\ &= \sup_Z |h| m(Z) \\ &= \sup_Z |h| \cdot 0 \\ &= 0,\end{aligned}$$

which implies

$$\int_a^b h = 0,$$

as desired. □

3. (Chapter 2, Problem 15) Let φ be a simple function defined on $[a,b]$.

   (a) Show that φ is measurable on $[a,b]$.

Proof. Recall that the simple function is defined

$$\varphi(x) = \sum_{k=1}^{n} a_k \chi_{E_k}(x)$$

for all $x \in [a, b]$, where a_k are constants and $E_k \subseteq [a, b]$ are pairwise disjoint Lebesgue measurable sets. Note that this means we can write $[a, b]$ as the disjoint union

$$[a, b] = \bigcup_{k=1}^{n} E_k,$$

which in particular implies that, for any $s \in \mathbb{R}$, we have

$$\begin{aligned}
\{x \in [a, b] : \varphi(x) > s\} &= \bigcup_{k=1}^{n} \{x \in E_k : \varphi(x) > s\} \\
&= \bigcup_{k=1}^{n} \{x \in E_k : a_k \chi_{E_k}(x) > s\} \\
&= \bigcup_{k=1}^{n} \{x \in E_k : a_k > s\},
\end{aligned}$$

where, depending on the value of s, each $\{x \in E_k : a_k > s\}$ is either E_k or $\varnothing$ (both of which are of course measurable sets), and that is because a_k is constant. In any case, $\bigcup_{k=1}^{n} \{x \in E_k : a_k > s\}$ can be one either the entire finite union for $k = 1, \dots, n$, or any subset of this finite union (including the empty set). Any union of measurable sets is measurable, according to some proposition of Nelson. So we conclude that $\{x \in [a, b] : \varphi(x) > s\}$ is a measurable set, and so φ is measurable on $[a, b]$. □

(b) Show that φ is Lebesgue integrable on $[a, b]$. Use the definition of the Lebesgue integral to compute $\int_a^b \varphi$.

Proof. Let $\epsilon > 0$ be given, and choose the partition P consisting only of endpoints of E_k for $k = 1, \dots, n$. Then we can write

$$\begin{aligned}
U(\varphi, P) - L(\varphi, P) &= \sum_{k=1}^{n} (\sup_{x \in E_k} \varphi) m(E_k) - \sum_{k=1}^{n} (\inf_{x \in E_k} \varphi) m(E_k) \\
&= \sum_{k=1}^{n} a_k m(E_k) - \sum_{k=1}^{n} a_k m(E_k) \\
&= 0 \\
&< \epsilon,
\end{aligned}$$

and so φ is Lebesgue integrable on $[a, b]$. Furthermore, assuming Exercise 19, we have

$$\begin{aligned}
\int_a^b \varphi &= \int_a^b \sum_{k=1}^{n} a_k \chi_{E_k} \\
&= \sum_{k=1}^{n} a_k \int_a^b \chi_{E_k} \\
&= \sum_{k=1}^{n} a_k \left(\int_{E_k} \chi_{E_k} + \int_{[a,b] \setminus E_k} \chi_{E_k} \right) \\
&= \sum_{k=1}^{n} a_k \left(\int_{E_k} 1 + \int_{[a,b] \setminus E_k} 0 \right) \\
&= \boxed{\sum_{k=1}^{n} a_k m(E_k)},
\end{aligned}$$

as desired. □

4. (Chapter 2, Problem 16) Let $f \in \mathcal{L}[a, b]$. Show that if g is a bounded measurable function, then $fg \in \mathcal{L}[a, b]$.

Proof. We will assume without proof that $f \in \mathcal{L}([a, b])$ implies $|f| \in \mathcal{L}([a, b])$, which means we have

$$\int_a^b |f| < \infty.$$

Since g is bounded, there exists $M > 0$ satisfying $|g(x)| \leq M$ for all $x \in [a, b]$. So we have

$$\begin{aligned} \left| \int_a^b fg \right| &\leq \int_a^b |f||g| \\ &\leq \int_a^b |f|M \\ &= \int_a^b |f| \\ &< \infty, \end{aligned}$$

meaning that we have $fg \in \mathcal{L}([a, b])$. ◻

5. (Chapter 2, Problem 17) Prove or give a counterexample: If $f, g \in \mathcal{L}[a, b]$, then $fg \in \mathcal{L}[a, b]$.

Proof. We will give a counterexample. Define f, g by $f(x) = g(x) = \frac{1}{\sqrt{x}}$. Then we have $f, g \in \mathcal{L}([a, b])$, according to Example 2.3.3 of Nelson. But we also have $f(x)g(x) = \frac{1}{\sqrt{x}}\frac{1}{\sqrt{x}} = \frac{1}{x}$, which means $fg \notin \mathcal{L}([a, b])$, according to Example 2.3.2 of Nelson. ◻

6. (Chapter 2, Problem 19) Let $f \in \mathcal{L}[a, b]$ and A and B be measurable subsets of $[a, b]$.

   (a) If $A \cap B = \emptyset$, show that
   $$\int_{A \cup B} f = \int_A f + \int_B f$$
   Proof. With $A \cap B = \emptyset$, we have $\chi_{A \cup B} = \chi_A + \chi_B$, which implies
   $$\begin{aligned} \int_{A\cup B} f &= \int_{A\cup B} f\chi_{A\cup B} \\ &= \int_{A\cup B} f(\chi_A + \chi_B) \\ &= \int_{A\cup B} (f\chi_A + f\chi_B) \\ &= \int_{A\cup B} f\chi_A + \int_{A\cup B} f\chi_B \\ &= \int_A f\chi_A + \int_B f\chi_B \\ &= \int_A f + \int_B f, \end{aligned}$$
   as desired. ◻

   (b) State and prove a result for the case that $A \cap B \neq \emptyset$.

   Proof. Since $A \cap B \neq \emptyset$, we have $\chi_{A\cup B} < \chi_A + \chi_B$, which implies
   $$\begin{aligned} \left|\int_{A\cup B} f\right| &\leq \int_{A\cup B} |f| \\ &= \int_{A\cup B} |f|\chi_{A\cup B} \\ &< \int_{A\cup B} |f|(\chi_A + \chi_B) \\ &= \int_{A\cup B} (|f|\chi_A + |f|\chi_B) \\ &= \int_{A\cup B} |f|\chi_A + \int_{A\cup B} |f|\chi_B \\ &= \int_A |f|\chi_A + \int_B |f|\chi_B \\ &= \int_A |f| + \int_B |f|, \end{aligned}$$
   as desired. ◻

   (c) What can you conclude if $A = [a, c]$ and $B = [c, b]$ for some $c \in (a, b)$?

Answer. We would have

$$\begin{aligned}
\int_a^b f &= \int_{[a,c)\cup\{c\}\cup(c,b]} f \\
&= \int_{[a,c)} f\chi_{[a,b)} + \int_{\{c\}} f\chi_{\{c\}} + \int_{(c,b]} f\chi_{(c,b]} \\
&= \int_{[a,c)} f + 0 + \int_{(c,b]} f \\
&= \int_{[a,c)} f + \int_{(c,b]} f,
\end{aligned}$$

as desired. □