MATH 165B discussion — Ryan Ta
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Solutions to suggested homework problems from
Complex Variables and Applications, Ninth Edition by James Brown and Ruel Churchill
Homework 1: Section 68, Exercises 1, 2, 3, 4, 5, 6

68.1. Find the Laurent series that represents the function

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$$

on the domain $0 < |z| < \infty$.

Solution. We have

$$\begin{aligned}
f(z) &= z^2 \sin\left(\frac{1}{z^2}\right) \\
&= z^2 \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{z^2})^{2n+1}}{(2n+1)!} \\
&= z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{(z^2)^{2n+1}} \\
&= z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{(z^2)^{2n} z^2} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{(z^2)^{2n}} \\
&= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{z^{4n}}}.
\end{aligned}$$

□

68.2. Find a representation for the function

$$f(z) = \frac{1}{1+z} = \frac{1}{z} \frac{1}{1+\frac{1}{z}}$$

in negative powers of z that is valid when $1 < |z| < \infty$.

Solution. We have

$$\begin{aligned}
f(z) &= \frac{1}{z}\frac{1}{1+\frac{1}{z}} \\
&= \frac{1}{z}\frac{1}{1-(-\frac{1}{z})} \\
&= \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{z^n} \\
&= \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^n}},
\end{aligned}$$

where for the second-to-last equality we made the substitution $n \mapsto n-1$, and where for the last equality we used $(-1)^{n-1} = (-1)^{n-1}\cdot 1 = (-1)^{n-1}(-1)^2 = (-1)^{(n-1)+2} = (-1)^{n+1}$. □

68.3. Find the Laurent series that represents the function

$$f(z) = \frac{1}{z(1+z^2)} = \frac{1}{z}\frac{1}{1+z^2}$$

when $1 < |z| < \infty$.

Solution. We have

$$\begin{aligned}
f(z) &= \frac{1}{z}\frac{1}{1+z^2} \\
&= \frac{1}{z}\left(\frac{1}{z^2}\frac{1}{\frac{1}{z^2}+1}\right) \\
&= \frac{1}{z^3}\frac{1}{1-(-\frac{1}{z^2})} \\
&= \frac{1}{z^3}\sum_{n=0}^{\infty}\left(-\frac{1}{z^2}\right)^n \\
&= \frac{1}{z^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n}} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+3}} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{z^{2(n-1)+3}} \\
&= \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}},
\end{aligned}$$

where for the second-to-last equality we made the substitution $n \mapsto n-1$, and where for the last equality we used $(-1)^{n-1} = (-1)^{n-1} \cdot 1 = (-1)^{n-1}(-1)^2 = (-1)^{(n-1)+2} = (-1)^{n+1}$. □

68.4. Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)}$$

and specify those regions in which those expansions are valid.

Solution. For all $0 < |z| < 1$, we have

$$\begin{aligned}
f(z) &= \frac{1}{z^2(1-z)} \\
&= \frac{1}{z^2}\frac{1}{1-z} \\
&= z^{-2}\sum_{n=0}^{\infty} z^n \\
&= \sum_{n=0}^{\infty} z^{n-2} \\
&= z^{0-2} + z^{1-2} + \sum_{n=2}^{\infty} z^{n-2} \\
&= \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} z^{(n+2)-2} \\
&= \boxed{\sum_{n=0}^{\infty} z^n + \frac{1}{z} + \frac{1}{z^2}},
\end{aligned}$$

where for the second-to-last equality we made the substitution $n \mapsto n+2$.

For all $1 < |z| < \infty$, we have

$$\begin{aligned} f(z) &= \frac{1}{z^2(1-z)} \\ &= \frac{1}{z^2}\frac{1}{1-z} \\ &= \frac{1}{z^2}\left(\frac{1}{z}\frac{1}{\frac{1}{z}-1}\right) \\ &= \frac{1}{z^2}\left(-\frac{1}{z}\frac{1}{1-\frac{1}{z}}\right) \\ &= -\frac{1}{z^3}\frac{1}{1-\frac{1}{z}} \\ &= -\frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \\ &= -\frac{1}{z^3}\sum_{n=0}^{\infty}\frac{1}{z^n} \\ &= -\sum_{n=0}^{\infty}\frac{1}{z^{n+3}} \\ &= -\sum_{n=3}^{\infty}\frac{1}{z^{(n-3)+3}} \\ &= \boxed{-\sum_{n=3}^{\infty}\frac{1}{z^n}}, \end{aligned}$$

where for the second-to-last equality we made the substitution $n \mapsto n-3$. □

68.5. The function

$$f(z) = -\frac{1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

which has two singular points $z = 1$ and $z = 2$, is analytic in the domains

$$D_1 : |z| < 1, \quad D_2 : 1 < |z| < 2, \quad D_3 : 2 < |z| < \infty.$$

Find the series representation in powers of z for $f(z)$ in each of those domains.

Solution. For all $|z| < 1$, we have

$$\begin{aligned}
f(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\
&= -\frac{1}{1-z} + \frac{1}{2-z} \\
&= -\frac{1}{1-z} + \frac{1}{2}\frac{1}{1-\frac{z}{2}} \\
&= -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \\
&= -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} \\
&= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \\
&= \sum_{n=0}^{\infty}\left(-1 + \frac{1}{2^{n+1}}\right) z^n \\
&= \boxed{\sum_{n=0}^{\infty}(2^{-n-1} - 1)z^n}.
\end{aligned}$$

For all $1 < |z| < 2$, we have

$$\begin{aligned}
f(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\
&= \frac{1}{z-1} + \frac{1}{2-z} \\
&= \frac{1}{z}\frac{1}{1-\frac{1}{z}} + \frac{1}{2}\frac{1}{1-\frac{z}{2}} \\
&= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \\
&= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} \\
&= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \\
&= \sum_{n=1}^{\infty}\frac{1}{z^{(n-1)+1}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \\
&= \boxed{\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty}\frac{1}{z^n}},
\end{aligned}$$

where for the second-to-last equality we made the substitution $n \mapsto n - 1$.

For all $2 < |z| < \infty$, we have

$$\begin{aligned}
f(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\
&= \frac{1}{z}\frac{1}{1-\frac{1}{z}} - \frac{1}{z}\frac{1}{1-\frac{2}{z}} \\
&= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n \\
&= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{z}\sum_{n=0}^{\infty}\frac{2^n}{z^n} \\
&= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} \\
&= \sum_{n=0}^{\infty}\frac{1-2^n}{z^{n+1}} \\
&= \sum_{n=1}^{\infty}\frac{1-2^{n-1}}{z^{(n-1)+1}} \\
&= \boxed{\sum_{n=1}^{\infty}\frac{1-2^{n-1}}{z^n}},
\end{aligned}$$

where for the second-to-last equality we made the substitution $n \mapsto n-1$. □

68.6. Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

Solution. For convenience, let $w := z-1$. Then our domain is $0 < |w| < 2$, and so, by employing the method of partial fractions, we obtain

$$\begin{aligned}
\frac{z}{(z-1)(z-3)} &= \frac{(z-1)+1}{(z-1)((z-1)-2)} \\
&= \frac{w+1}{w(w-2)} \\
&= \frac{\frac{3}{2}}{w-2} + \frac{-\frac{1}{2}}{w} \\
&= \frac{3}{2}\frac{1}{w-2} - \frac{1}{2w} \\
&= \frac{3}{2}\left(\frac{1}{2}\frac{1}{\frac{w}{2}-1}\right) - \frac{1}{2w} \\
&= \frac{3}{2}\left(-\frac{1}{2}\frac{1}{1-\frac{w}{2}}\right) - \frac{1}{2w} \\
&= -\frac{3}{2^2}\frac{1}{1-\frac{w}{2}} - \frac{1}{2w} \\
&= -\frac{3}{2^2}\sum_{n=0}^{\infty}\left(\frac{w}{2}\right)^n - \frac{1}{2w} \\
&= -\frac{3}{2^2}\sum_{n=0}^{\infty}\frac{w^n}{2^n} - \frac{1}{2w} \\
&= -3\sum_{n=0}^{\infty}\frac{w^n}{2^{n+2}} - \frac{1}{2w} \\
&= \boxed{-3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}}.
\end{aligned}$$

□