MATH 165B discussion — Ryan Ta
University of California, Riverside — Spring 2022

Solutions to suggested homework problems from
Complex Variables and Applications, Ninth Edition by James Brown and Ruel Churchill
Homework 2: Section 77, Exercises 1(a)(b)(c), 2(a)(b)(c)(d), 3, 4(a)(b)(c), 6

77.1. Find the residue at the singularity $z = 0$ of the function

(a) $\dfrac{1}{z+z^2}$.

Solution. We have

$$\begin{aligned}
\frac{1}{z+z^2} &= \frac{1}{z(1+z)} \\
&= \frac{1}{z}\frac{1}{1-(-z)} \\
&= \frac{1}{z}\sum_{n=0}^{\infty}(-z)^n \\
&= z^{-1}\sum_{n=0}^{\infty}(-1)^n z^n \\
&= \sum_{n=0}^{\infty}(-1)^n z^{n-1} \\
&= (-1)^0 z^{0-1} + \sum_{n=1}^{\infty}(-1)^n z^{n-1} \\
&= \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^n z^{n-1}.
\end{aligned}$$

The residue at the singularity $z = 0$ is the coefficient of $\dfrac{1}{z}$, which is $\boxed{1}$. □

(b) $z\cos\left(\dfrac{1}{z}\right)$.

Solution. We have

$$\begin{aligned}
z\cos\left(\frac{1}{z}\right) &= z\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{z})^{2n}}{(2n)!} \\
&= z\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{-2n} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{1-2n} \\
&= \frac{(-1)^0}{(2(0))!}z^{1-2(0)} + \frac{(-1)^1}{(2(1))!}z^{1-2(1)} + \sum_{n=2}^{\infty}\frac{(-1)^n}{(2n)!}z^{1-2n} \\
&= z - \frac{1}{2z} + \sum_{n=2}^{\infty}\frac{(-1)^n}{(2n)!}z^{1-2n}.
\end{aligned}$$

The residue at the singularity $z = 0$ is the coefficient of $\frac{1}{z}$, which is $\boxed{-\frac{1}{2}}$. □

(c) $\dfrac{z - \sin(z)}{z}$.

Solution. We have

$$\begin{aligned}
\frac{z - \sin(z)}{z} &= \frac{z - \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}}{z} \\
&= \frac{z(1 - \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!})}{z} \\
&= 1 - \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!} \\
&= 1 - \left((-1)^0 \frac{z^{2(0)}}{(2(0))!} + \sum_{n=1}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!}\right) \\
&= 1 - \left(\frac{1}{2} + \sum_{n=1}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!}\right) \\
&= \frac{1}{2} - \sum_{n=1}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!} \\
&= \frac{0}{z} + \frac{1}{2} - \sum_{n=1}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!}.
\end{aligned}$$

The residue at the singularity $z = 0$ is the coefficient of $\frac{1}{z}$, which is $\boxed{0}$. □

77.2. Use Cauchy's residue theorem (Section 76) to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

(a) $\dfrac{\exp(-z)}{z^2}$.

Solution. We have

$$\begin{aligned}
\frac{\exp(-z)}{z^2} &= \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{(-z)^n}{n!} \\
&= z^{-2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!} z^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!} z^{n-2} \\
&= \frac{(-1)^0}{0!} z^{0-2} + \frac{(-1)^1}{1!} z^{1-2} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!} z^{n-2} \\
&= \frac{1}{z^2} - \frac{1}{z} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!} z^{n-2}.
\end{aligned}$$

The residue at the singularity $z = 0$ is the coefficient of $\frac{1}{z}$, which is

$$\operatorname*{Res}_{z=0} \frac{\exp(-z)}{z^2} = -1.$$

By Cauchy's residue theorem, we conclude

$$\begin{aligned}
\int_{|z|=3} \frac{\exp(-z)}{z^2}\, dz &= 2\pi i \operatorname*{Res}_{z=0} \frac{\exp(-z)}{z^2} \\
&= 2\pi i(-1) \\
&= \boxed{-2\pi i}.
\end{aligned}$$

□

(b) $\dfrac{\exp(-z)}{(z-1)^2}$.

Solution. We have

$$\begin{aligned}
\frac{\exp(-z)}{(z-1)^2} &= \frac{\exp(-1-z+1)}{(z-1)^2} \\
&= \frac{\exp(-1-(z-1))}{(z-1)^2} \\
&= \frac{\exp(-1)\exp(-(z-1))}{(z-1)^2} \\
&= \frac{e^{-1}}{(z-1)^2} \sum_{n=0}^{\infty} \frac{(-(z-1))^n}{n!} \\
&= \frac{1}{e}(z-1)^{-2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(z-1)^n \\
&= \frac{1}{e} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(z-1)^{n-2} \\
&= \frac{1}{e}\left(\frac{(-1)^0}{0!}(z-1)^{0-2} + \frac{(-1)^1}{1!}(z-1)^{1-2} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!}(z-1)^{n-2}\right) \\
&= \frac{1}{e}\left(\frac{1}{(z-1)^2} - \frac{1}{z-1} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!}(z-1)^{n-2}\right) \\
&= \frac{1}{e(z-1)^2} - \frac{1}{e(z-1)} + \sum_{n=2}^{\infty} \frac{(-1)^n}{en!}(z-1)^{n-2}.
\end{aligned}$$

The residue at the singularity $z = 0$ is the coefficient of $\frac{1}{z}$, which is

$$\operatorname*{Res}_{z=0} \frac{\exp(-z)}{(z-1)^2} = -\frac{1}{e}.$$

By Cauchy's residue theorem, we conclude

$$\begin{aligned}
\int_{|z|=3} z^2 \exp\left(\frac{1}{z}\right) dz &= 2\pi i \operatorname*{Res}_{z=0} z^2 \exp\left(\frac{1}{z}\right) \\
&= 2\pi i \left(-\frac{1}{e}\right) \\
&= \boxed{-\frac{2\pi i}{e}}.
\end{aligned}$$

□

(c) $z^2 \exp\left(\frac{1}{z}\right)$.

Solution. We have

$$\begin{aligned}
z^2 \exp\left(\frac{1}{z}\right) &= z^2 \sum_{n=0}^{\infty} \frac{(\frac{1}{z})^n}{n!} \\
&= z^2 \sum_{n=0}^{\infty} \frac{1}{n} z^{-n} \\
&= \sum_{n=0}^{\infty} \frac{1}{n} z^{2-n} \\
&= \frac{1}{0!} z^{2-0} + \frac{1}{1!} z^{2-1} + \frac{1}{2!} z^{2-2} + \frac{1}{3!} z^{2-3} + \sum_{n=4}^{\infty} \frac{1}{n} z^{2-n} \\
&= z^2 + z + \frac{1}{2} + \frac{1}{6z} + \sum_{n=4}^{\infty} \frac{1}{n} z^{2-n}.
\end{aligned}$$

The residue at the singularity $z = 1$ is the coefficient of $\frac{1}{z-1}$, which is

$$\operatorname*{Res}_{z=0} \frac{\exp(-z)}{(z-1)^2} = \frac{1}{6}.$$

By Cauchy's residue theorem, we conclude

$$\begin{aligned}
\int_{|z|=3} \frac{\exp(-z)}{(z-1)^2} dz &= 2\pi i \operatorname*{Res}_{z=1} \frac{\exp(-z)}{(z-1)^2} \\
&= 2\pi i \left(\frac{1}{6}\right) \\
&= \boxed{\frac{\pi i}{3}}.
\end{aligned}$$

□

(d) $\dfrac{z+1}{z^2 - 2z}$.

Solution. We can split the integrand into partial fractions:

$$\frac{z+1}{z^2-2z} = \frac{3}{2(z-2)} - \frac{1}{2z}.$$

By the linearity of the contour integral, we have

$$\begin{aligned}\int_{|z|=3} \frac{z+1}{z^2-2z}\,dz &= \int_{|z|=3} \frac{3}{2(z-2)} - \frac{1}{2z}\,dz \\ &= \frac{3}{2}\int_{|z|=3} \frac{1}{z-2}\,dz - \frac{1}{2}\int_{|z|=3} \frac{1}{z}\,dz.\end{aligned}$$

Now, by Cauchy's residue theorem, we obtain

$$\begin{aligned}\int_{|z|=3} \frac{1}{z-2}\,dz &= 2\pi i \operatorname*{Res}_{z=2} \frac{1}{z-2} \\ &= 2\pi i(1) \\ &= 2\pi i\end{aligned}$$

and

$$\begin{aligned}\int_{|z|=3} \frac{1}{z}\,dz &= 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z} \\ &= 2\pi i(1) \\ &= 2\pi i.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\int_{|z|=3} \frac{z+1}{z^2-2z}\,dz &= \frac{3}{2}\int_{|z|=3} \frac{1}{z-2}\,dz - \frac{1}{2}\int_{|z|=3} \frac{1}{z}\,dz \\ &= \frac{3}{2}(2\pi i) - \frac{1}{2}(2\pi i) \\ &= 3\pi i - \pi i \\ &= \boxed{2\pi i}.\end{aligned}$$

□

77.3. In the example in Section 76, two residues were used to evaluate the integral

$$\int_C \frac{4z-5}{z(z-1)}\,dz,$$

where C is the positively oriented circle $|z| = 2$. Evaluate this integral once again by using the theorem in Section 77 and finding only one residue.

Solution. By setting $f(z) = \dfrac{4z-5}{z(z-1)}$, we have

$$\begin{aligned}
f\left(\frac{1}{z}\right) &= \frac{4(\frac{1}{z})-5}{\frac{1}{z}(\frac{1}{z}-1)} \\
&= \frac{4(\frac{1}{z})-5}{\frac{1}{z}(\frac{1}{z}-1)}\frac{z^2}{z^2} \\
&= \frac{4z-5z^2}{1-z} \\
&= \frac{z(4-5z)}{1-z},
\end{aligned}$$

and so we have

$$\begin{aligned}
\frac{1}{z^2}f\left(\frac{1}{z}\right) &= \frac{1}{z^2}\frac{z(4-5z)}{1-z} \\
&= \frac{4-5z}{z}\frac{1}{1-z} \\
&= \frac{4-5z}{z}\sum_{n=0}^{\infty} z^n \\
&= (4-5z)z^{-1}\sum_{n=0}^{\infty} z^n \\
&= (4-5z)\sum_{n=0}^{\infty} z^{n-1} \\
&= (4-5z)\left(z^{0-1}+\sum_{n=1}^{\infty} z^{n-1}\right) \\
&= (4-5z)\left(\frac{1}{z}+\sum_{n=1}^{\infty} z^{n-1}\right) \\
&= (4-5z)\frac{1}{z}+(4-5z)\sum_{n=1}^{\infty} z^{n-1} \\
&= \frac{4}{z}-5+\sum_{n=1}^{\infty}(4-5z)z^{n-1} \\
&= \frac{4}{z}-5+\sum_{n=1}^{\infty}(4z^{n-1}-5z^n).
\end{aligned}$$

The residue at the singularity $z = 0$ is the coefficient of $\dfrac{1}{z}$, which is

$$\operatorname*{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right] = 4.$$

By the theorem in Section 77, we obtain

$$\begin{aligned}\int_{|z|=2} \frac{4z-5}{z(z-1)}\,dz &= 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] \\ &= 2\pi i(4) \\ &= \boxed{8\pi i}.\end{aligned}$$

□

77.4. Use the theorem in Section 77, involving a single residue, to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense:

(a) $\dfrac{z^5}{1-z^3}$.

Solution. By setting $f(z) = \dfrac{z^5}{1-z^3}$, we have

$$\begin{aligned}f\left(\frac{1}{z}\right) &= \frac{(\frac{1}{z})^5}{1-(\frac{1}{z})^3} \\ &= \frac{\frac{1}{z^5}}{1-\frac{1}{z^3}} \\ &= \frac{\frac{1}{z^5}}{1-\frac{1}{z^3}}\frac{z^3}{z^3} \\ &= \frac{\frac{1}{z^2}}{z^3-1} \\ &= \frac{1}{z^2(z^3-1)},\end{aligned}$$

and so we have

$$
\begin{aligned}
\frac{1}{z^2} f\left(\frac{1}{z}\right) &= \frac{1}{z^2} \frac{1}{z^2(z^3-1)} \\
&= \frac{1}{z^4(z^3-1)} \\
&= -\frac{1}{z^4} \frac{1}{1-z^3} \\
&= -\frac{1}{z^4} \sum_{n=0}^{\infty} (z^3)^n \\
&= -z^{-4} \sum_{n=0}^{\infty} z^{3n} \\
&= -\sum_{n=0}^{\infty} z^{3n-4} \\
&= -\left(z^{3(0)-4} + z^{3(1)-4} + \sum_{n=2}^{\infty} z^{3n-4}\right) \\
&= -\left(\frac{1}{z^4} + \frac{1}{z} + \sum_{n=2}^{\infty} z^{3n-4}\right) \\
&= -\frac{1}{z^4} - \frac{1}{z} - \sum_{n=2}^{\infty} z^{3n-4}.
\end{aligned}
$$

The residue at the singularity $z = 0$ is the coefficient of $\frac{1}{z}$, which is

$$
\operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = -1.
$$

By the theorem in Section 77, we obtain

$$
\begin{aligned}
\int_{|z|=2} \frac{z^5}{1-z^3}\, dz &= 2\pi i \operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] \\
&= 2\pi i(-1) \\
&= \boxed{-2\pi i}.
\end{aligned}
$$

□

(b) $\frac{1}{1+z^2}$.

Solution. By setting $f(z) = \dfrac{1}{1+z^2}$, we have

$$\begin{aligned} f\left(\frac{1}{z}\right) &= \frac{1}{1+(\frac{1}{z})^2} \\ &= \frac{1}{1+\frac{1}{z^2}} \\ &= \frac{1}{1+\frac{1}{z^2}}\frac{z^2}{z^2} \\ &= \frac{z^2}{z^2+1} \\ &= \frac{z^2}{1+z^2}, \end{aligned}$$

and so we have

$$\begin{aligned} \frac{1}{z^2} f\left(\frac{1}{z}\right) &= \frac{1}{z^2}\frac{z^2}{1+z^2} \\ &= \frac{1}{1+z^2} \\ &= \frac{1}{1-(-z^2)} \\ &= \sum_{n=0}^{\infty}(-z^2)^n \\ &= \frac{0}{z} + \sum_{n=0}^{\infty}(-1)^n z^{2n} \end{aligned}$$

The residue at the singularity $z = 0$ is the coefficient of $\dfrac{1}{z}$, which is

$$\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = 0.$$

By the theorem in Section 77, we obtain

$$\begin{aligned} \int_{|z|=2} \frac{z^5}{1-z^3}\,dz &= 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] \\ &= 2\pi i(0) \\ &= \boxed{0}. \end{aligned}$$

□

(c) $\dfrac{1}{z}$.

Solution. By setting $f(z) = \dfrac{1}{z}$, we have

$$\begin{aligned} f\left(\frac{1}{z}\right) &= \frac{1}{\frac{1}{z}} \\ &= z, \end{aligned}$$

and so we have

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} z$$
$$= \frac{1}{z}.$$

The residue at the singularity $z = 0$ is the coefficient of $\frac{1}{z}$, which is

$$\operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = 1.$$

By the theorem in Section 77, we obtain

$$\begin{aligned}\int_{|z|=2} \frac{z^5}{1-z^3}\, dz &= 2\pi i \operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] \\ &= 2\pi i(1) \\ &= \boxed{2\pi i}.\end{aligned}$$

□

77.6. Suppose that a function f is analytic throughout the finite plane except for a finite number of singular points $z_1, z_2, \ldots, z_n$. Show that

$$\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

Proof. First, let us write

$$\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

as its more condensed version:

$$\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

Cauchy's residue theorem from Section 76 states:

$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

Equation (2) of Section 77 states:

$$\int_C f(z)\, dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z).$$

Therefore, we have

$$\begin{aligned}\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) + \operatorname*{Res}_{z=\infty} f(z) &= \frac{1}{2\pi i}\left(2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) - -2\pi i \operatorname*{Res}_{z=\infty} f(z)\right) \\ &= \frac{1}{2\pi i}\left(\int_C f(z)\, dz - \int_C f(z)\, dz\right) \\ &= \frac{1}{2\pi i} 0 \\ &= 0,\end{aligned}$$

as desired. □