MATH 165B discussion Ryan Ta
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Solutions to suggested homework problems from
Complex Variables and Applications, Ninth Edition by James Brown and Ruel Churchill
Homework 3: Section 79, Exercises 1(a)(b)(c)(d)(e), 2(b)(c), 3 and Section 81, Exercises 1(a)(b)(c)(d), 2(a)(b), 3(b), 4, 5, 7(a)

79.1. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a removable singular point, an essential singular point, or a pole:

(a) $z \exp\left(\frac{1}{z}\right)$

Solution. The singular point of $z \exp\left(\frac{1}{z}\right)$ occurs at $z = 0$. The Laurent series expansion about $z = 0$ of $z \exp\left(\frac{1}{z}\right)$ is

$$\begin{aligned}
z \exp\left(\frac{1}{z}\right) &= z \sum_{n=0}^{\infty} \frac{(\frac{1}{z})^n}{n!} \\
&= z \sum_{n=0}^{\infty} \frac{\frac{1}{z^n}}{n!} \\
&= z \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{z^{-n+1}}{n!} \\
&= \frac{z^{-0+1}}{0!} + \frac{z^{-1+1}}{1!} + \sum_{n=2}^{\infty} \frac{z^{-n+1}}{n!} \\
&= z + 1 + \sum_{n=2}^{\infty} \frac{z^{-n+1}}{n!}.
\end{aligned}$$

Since the principal part $\sum_{n=1}^{\infty} \frac{z^{-n+1}}{n!}$ consists of infinitely many terms, we determine that the point $z = 0$ is an essential singular point. □

(b) $\frac{z^2}{1+z}$

Solution. The singular point of $\frac{z^2}{1+z}$ occurs at $z = -1$. The Laurent series expansion about $z = -1$ of $\dfrac{z^2}{1+z}$ is

$$\begin{aligned}\frac{z^2}{1+z} &= \frac{((1+z)-1)^2}{1+z} \\ &= \frac{(1+z)^2 + 2(1+z) + 1}{1+z} \\ &= \frac{(1+z)^2}{1+z} + \frac{2(1+z)}{1+z} + \frac{1}{1+z} \\ &= (1+z) + 1 + \frac{1}{1+z} \\ &= 2 + z + \frac{1}{1+z}.\end{aligned}$$

Since the principal part $\dfrac{1}{1+z}$ consists of only one term, we determine that the point $z = -1$ is a pole of order $m = 1$. □

(c) $\dfrac{\sin z}{z}$

Solution. The singular point of $\dfrac{\sin z}{z}$ occurs at $z = 0$. The Laurent series expansion about $z = 0$ of $\dfrac{\sin z}{z}$ is

$$\begin{aligned}\frac{\sin z}{z} &= \frac{\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}}{z} \\ &= z^{-1} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n+1)!}.\end{aligned}$$

Since every term of the principal part of this series is zero, we determine that the point $z = 0$ is a removable singular point. □

(d) $\dfrac{\cos z}{z}$

Solution. The singular point of $\dfrac{\cos z}{z}$ occurs at $z = 0$. The Laurent series expansion about $z = 0$ of $\dfrac{\cos z}{z}$ is

$$\begin{aligned}
\frac{\cos z}{z} &= \frac{\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}}{z} \\
&= z^{-1} \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} \frac{z^{2n-1}}{(2n)!} \\
&= \frac{z^{2(0)-1}}{(2(0))!} + \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n)!} \\
&= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n-1}}{(2n)!}.
\end{aligned}$$

Since the principal part $\dfrac{1}{z}$ consists of only one term, we determine that the point $z = 0$ is a pole of order $m = 1$. □

79.2. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

(b) $\dfrac{1 - \exp(2z)}{z^4}$

Solution. The singular point of $\dfrac{1 - \exp(2z)}{z^4}$ occurs at $z = 0$. The Laurent series expansion about $z = 0$ of $\dfrac{1 - \exp(2z)}{z^4}$ is

$$\begin{aligned}
\frac{1 - \exp(2z)}{z^4} &= \frac{1 - \sum_{n=0}^{\infty} \frac{(2z)^n}{n!}}{z^4} \\
&= \frac{1 - (\frac{(2z)^0}{0!} + \sum_{n=1}^{\infty} \frac{(2z)^n}{n!})}{z^4} \\
&= \frac{1 - (1 + \sum_{n=1}^{\infty} \frac{2^n z^n}{n!})}{z^4} \\
&= \frac{- \sum_{n=1}^{\infty} \frac{2^n z^n}{n!}}{z^4} \\
&= -z^{-4} \sum_{n=1}^{\infty} \frac{2^n z^n}{n!} \\
&= -\sum_{n=1}^{\infty} \frac{2^n z^{n-4}}{n!} \\
&= -\left(\frac{2^1 z^{1-4}}{1!} + \frac{2^2 z^{2-4}}{2!} + \frac{2^3 z^{3-4}}{3!} + \sum_{n=4}^{\infty} \frac{2^n z^{n-4}}{n!} \right) \\
&= -\frac{2}{z^3} - \frac{2}{z^2} - \frac{4}{3z} - \sum_{n=4}^{\infty} \frac{2^n z^{n-4}}{n!}.
\end{aligned}$$

Since the principal part $-\dfrac{2}{z^3}-\dfrac{2}{z^2}-\dfrac{4}{3z}$ consists of only a finite number of terms, we determine that the point $z=0$ is a pole of order $m=3$. Furthermore, the residue at the singular point $z=0$ is the coefficient of $\dfrac{1}{z}$, which is

$$\operatorname*{Res}_{z=0}\frac{1-\exp(2z)}{z^4}=-\frac{4}{3}.$$

□

(c) $\dfrac{\exp(2z)}{(z-1)^2}$

Solution. The singular point of $\dfrac{\exp(2z)}{(z-1)^2}$ occurs at $z=0$. The Laurent series expansion about $z=0$ of $\dfrac{\exp(2z)}{(z-1)^2}$ is

$$\begin{aligned}
\frac{\exp(2z)}{(z-1)^2} &= \frac{\exp(2((z-1)+1))}{(z-1)^2}\\
&= \frac{\exp(2(z-1)+2)}{(z-1)^2}\\
&= \frac{\exp(2(z-1))\exp(2)}{(z-1)^2}\\
&= \frac{e^2}{(z-1)^2}\sum_{n=0}^{\infty}\frac{(2(z-1))^n}{n!}\\
&= \frac{e^2}{(z-1)^2}\sum_{n=0}^{\infty}\frac{2^n(z-1)^n}{n!}\\
&= e^2\sum_{n=0}^{\infty}\frac{2^n}{n!}(z-1)^{n-2}\\
&= e^2\left(\frac{2^0}{0!}(z-1)^{0-2}+\frac{2^1}{1!}(z-1)^{1-2}+\sum_{n=2}^{\infty}\frac{2^n}{n!}(z-1)^{n-2}\right)\\
&= e^2\left(\frac{1}{(z-1)^2}+\frac{2}{z-1}+\sum_{n=2}^{\infty}\frac{2^n}{n!}(z-1)^{n-2}\right)\\
&= \frac{e^2}{(z-1)^2}+\frac{2e^2}{z-1}+\sum_{n=2}^{\infty}\frac{2^ne^2}{n!}(z-1)^{n-2}.
\end{aligned}$$

Since the principal part $\dfrac{e^2}{(z-1)^2}+\dfrac{2e^2}{z-1}$ consists of only a finite number of terms, we determine that the point $z=1$ is a pole of order $m=2$. Furthermore, the residue at the singular point $z=1$ is the coefficient of $\dfrac{1}{z-1}$, which is

$$\operatorname*{Res}_{z=0}\frac{\exp(2z)}{(z-1)^2}=2e^2.$$

□

79.3. Suppose that a function f is analytic at z_0, and write $g(z) = \dfrac{f(z)}{z - z_0}$. Show that

(a) if $f(z_0) \neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$;

(b) if $f(z_0) = 0$, then z_0 is a removable singular point of g.

Suggestion: As pointed out in Section 62, there is a Taylor series for $f(z)$ about z_0 since f is analytic there. Start each part of this exercise by writing out a few terms of that series.

Proof. By the theorem in Section 62, since f is analytic at z_0, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $a_n = \dfrac{f^{(n)}(z_0)}{n!}$. As a result, we have

$$\begin{aligned} g(z) &= \frac{f(z)}{z - z_0} \\ &= (z - z_0)^{-1} f(z) \\ &= (z - z_0)^{-1} \sum_{n=0}^{\infty} a_n (z - z_0)^n \\ &= \sum_{n=0}^{\infty} a_n (z - z_0)^{n-1} \\ &= a_0 (z - z_0)^{-1} + \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1} \\ &= \frac{a_0}{z - z_0} + \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1}. \end{aligned}$$

(a) If $f(z_0) \neq 0$, then we have

$$\begin{aligned} a_0 &= \frac{f^{(0)}(z_0)}{(z - z_0)^0} \\ &= \frac{f(z_0)}{1} \\ &= f(z_0), \end{aligned}$$

and so we have

$$\begin{aligned} g(z) &= \frac{a_0}{z - z_0} + \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1} \\ &= \frac{f(z_0)}{z - z_0} + \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1}, \end{aligned}$$

which shows that z_0 is a simple pole of g. Furthermore, the residue at the singular point $z = z_0$ is the coefficient of $\dfrac{1}{z - z_0}$, which is

$$\operatorname*{Res}_{z=0} f(z) = f(z_0).$$

If $f(z_0) = 0$, then we have

$$a_0 = \frac{f^{(0)}(z_0)}{(z - z_0)^0} = \frac{0}{1} = 0,$$

and so we have

$$\begin{aligned} g(z) &= \frac{a_0}{z - z_0} + \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1} \\ &= \frac{0}{z - z_0} + \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1} \\ &= \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1}, \end{aligned}$$

which shows that z_0 is a removable singular point of g. □

81.1. In each case, show that any singular point of the function is a pole. Determine the order m of each pole, and find the corresponding residue B.

(a) $\dfrac{z+1}{z^2+9}$

Solution. The singularities of $\dfrac{z+1}{z^2+9}$ are $z = \pm 3i$, both of which are poles of order 1. For the pole $z = 3i$ of order $m = 1$, we can write

$$\frac{z+1}{z^2+9} = \frac{z+1}{(z+3i)(z-3i)} = \frac{\phi(z)}{z - 3i},$$

where we define

$$\phi(z) = \frac{z+1}{z+3i},$$

and so the residue of $\frac{z+1}{z^2+9}$ at $z = 3i$ is

$$\begin{aligned}
\operatorname*{Res}_{z=3i} \frac{z+1}{z^2+9} &= \phi(3i) \\
&= \frac{3i+1}{3i+3i} \\
&= \frac{3i+1}{6i} \\
&= \frac{3i+1}{6i}\frac{6i}{6i} \\
&= \frac{18i^2+6i}{36i^2} \\
&= \frac{18(-1)+6i}{36(-1)} \\
&= \frac{-18+6i}{-36} \\
&= \frac{-6(3-i)}{-36} \\
&= \boxed{\frac{3-i}{6}}.
\end{aligned}$$

For the pole $z = -3i$ of order $m = 1$, we can write

$$\begin{aligned}
\frac{z+1}{z^2+9} &= \frac{z+1}{(z+3i)(z-3i)} \\
&= \frac{\phi(z)}{z+3i},
\end{aligned}$$

where we define

$$\phi(z) = \frac{z+1}{z-3i},$$

and so the residue of $\frac{z+1}{z^2+9}$ at $z = 3i$ is

$$\begin{aligned}
\operatorname*{Res}_{z=-3i} \frac{z+1}{z^2+9} &= \phi(-3i) \\
&= \frac{-3i+1}{-3i-3i} \\
&= \frac{-3i+1}{-6i} \\
&= \frac{-3i+1}{-6i}\frac{6i}{6i} \\
&= \frac{-18i^2+6i}{-36i^2} \\
&= \frac{-18(-1)+6i}{-36(-1)} \\
&= \frac{18+6i}{36} \\
&= \frac{6(3+i)}{36} \\
&= \boxed{\frac{3+i}{6}}.
\end{aligned}$$

□

(b) $\dfrac{z^2+2}{z-1}$

Solution. The only singular point of $\dfrac{z^2+2}{z-1}$ is $z = 1$, which is a pole of order 1. We can write

$$\frac{z^2+2}{z-1} = \frac{\phi(z)}{z-1}$$

where we define

$$\phi(z) = z^2 + 2,$$

and so the residue of $\dfrac{z^2+2}{z-1}$ at $z = 1$ is

$$\begin{aligned}
\operatorname*{Res}_{z=1} \frac{z^2+2}{z-1} &= \phi(1) \\
&= 1^2 + 2 \\
&= 1 + 2 \\
&= \boxed{3}.
\end{aligned}$$

□

(c) $\left(\dfrac{z}{2z+1}\right)^3$

Solution. The only singular point of $\left(\frac{z}{2z+1}\right)^3$ is $z = -\frac{1}{2}$, which is a pole of order 3. We can write

$$\begin{aligned}
\left(\frac{z}{2z+1}\right)^3 &= \frac{z^3}{(2z+1)^3} \\
&= \frac{z^3}{(2(z+\frac{1}{2}))^3} \\
&= \frac{z^3}{2^3(z+\frac{1}{2})^3} \\
&= \frac{z^3}{8(z+\frac{1}{2})^3} \\
&= \frac{\phi(z)}{(z+\frac{1}{2})^3}
\end{aligned}$$

where we define

$$\phi(z) = \frac{z^3}{8},$$

whose second derivative is

$$\begin{aligned}
\phi''(z) &= \frac{d^2}{dz^2}\left(\frac{z^3}{8}\right) \\
&= \frac{d}{dz}\left(\frac{3z^2}{8}\right) \\
&= \frac{6z}{8} \\
&= \frac{3z}{4},
\end{aligned}$$

and so the residue of $\left(\frac{z}{2z+1}\right)^3$ at $z = -\frac{1}{2}$ is

$$\begin{aligned}
\operatorname*{Res}_{z=1}\left(\frac{z}{2z+1}\right)^3 &= \frac{\phi^{(3-1)}(-\frac{1}{2})}{(3-1)!} \\
&= \frac{\phi^{(2)}(-\frac{1}{2})}{2!} \\
&= \frac{\phi''(-\frac{1}{2})}{2} \\
&= \frac{\frac{3(-\frac{1}{2})}{4}}{2} \\
&= \frac{\frac{-\frac{3}{2}}{4}}{2} \\
&= \frac{1}{2}\frac{1}{4}\left(-\frac{3}{2}\right) \\
&= \boxed{-\frac{3}{16}}.
\end{aligned}$$

□

(d) $\dfrac{e^z}{z^2+\pi^2}$

Solution. The singularities of $\dfrac{e^z}{z^2+\pi^2}$ are $z = \pm\pi i$, both of which are poles of order 1. For the pole $z = \pi i$ of order $m = 1$, we can write

$$\begin{aligned} \frac{e^z}{z^2+\pi^2} &= \frac{e^z}{(z+\pi i)(z-\pi i)} \\ &= \frac{\phi(z)}{z-\pi i}, \end{aligned}$$

where we define

$$\phi(z) = \frac{e^z}{z+\pi i},$$

and so the residue of $\frac{e^z}{z^2+\pi^2}$ at $z = \pi i$ is

$$\begin{aligned} \operatorname*{Res}_{z=\pi i} \frac{e^z}{z^2+\pi^2} &= \phi(\pi i) \\ &= \frac{e^{\pi i}}{\pi i + \pi i} \\ &= \frac{\cos(\pi) + i\sin(\pi)}{2\pi i} \\ &= \frac{-1+i0}{2\pi i} \\ &= -\frac{1}{2\pi i} \\ &= -\frac{1}{2\pi i}\frac{i}{i} \\ &= -\frac{i}{2\pi i^2} \\ &= -\frac{i}{2\pi(-1)} \\ &= \boxed{\frac{i}{2\pi}}. \end{aligned}$$

For the pole $z = -\pi i$ of order $m = 1$, we can write

$$\begin{aligned} \frac{e^z}{z^2+\pi^2} &= \frac{e^z}{(z+\pi i)(z-\pi i)} \\ &= \frac{\phi(z)}{z+\pi i}, \end{aligned}$$

where we define

$$\phi(z) = \frac{e^z}{z-\pi i},$$

and so the residue of $\frac{e^z}{z^2+\pi^2}$ at $z = -\pi i$ is

$$\begin{aligned}
\operatorname*{Res}_{z=\pi i} \frac{e^z}{z^2+\pi^2} &= \phi(-\pi i) \\
&= \frac{e^{-\pi i}}{-\pi i - \pi i} \\
&= \frac{\cos(-\pi) + i\sin(-\pi)}{-2\pi i} \\
&= \frac{-1 + i0}{-2\pi i} \\
&= \frac{1}{2\pi i} \\
&= \frac{1}{2\pi i}\frac{i}{i} \\
&= \frac{i}{2\pi i^2} \\
&= \frac{i}{2\pi(-1)} \\
&= \boxed{-\frac{i}{2\pi}}.
\end{aligned}$$

□

81.2. Show that

(a) $\displaystyle\operatorname*{Res}_{z=-1} \frac{z^{\frac{1}{4}}}{z+1} = \frac{1+i}{\sqrt{2}} \quad (|z| > 0, 0 < \arg z < 2\pi)$

Solution. For the pole $z = -1$ of order $m = 1$, we can write

$$\frac{z^{\frac{1}{4}}}{z+1} = \frac{\phi(z)}{z - (-1)},$$

where we define

$$\phi(z) = z^{\frac{1}{4}},$$

and so the residue of $\frac{z^{\frac{1}{4}}}{z+1}$ at $z = -1$ is

$$\begin{aligned}
\operatorname*{Res}_{z=-1} \frac{z^{\frac{1}{4}}}{z+1} &= \phi(-1) \\
&= (-1)^{\frac{1}{4}} \\
&= (e^{i\pi})^{\frac{1}{4}} \\
&= e^{i\frac{\pi}{4}} \\
&= \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) \\
&= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \\
&= \boxed{\frac{1+i}{\sqrt{2}}}
\end{aligned}$$

□

(b) $\operatorname*{Res}_{z=i} \dfrac{\operatorname{Log} z}{(z^2+1)^2} = \dfrac{\pi + 2i}{8}$

Solution. For the pole $z = i$ of order $m = 2$, we can write

$$\begin{aligned}\frac{\operatorname{Log} z}{(z^2+1)^2} &= \frac{\operatorname{Log} z}{((z+i)(z-i))^2} \\ &= \frac{\operatorname{Log} z}{(z+i)^2(z-i)^2} \\ &= \frac{\phi(z)}{(z-i)^2},\end{aligned}$$

where we define

$$\phi(z) = \frac{\operatorname{Log} z}{(z+i)^2},$$

whose first derivative is

$$\begin{aligned}\phi'(z) &= \frac{d}{dz}\frac{\operatorname{Log} z}{(z+i)^2} \\ &= \frac{\frac{d}{dz}(\operatorname{Log} z)((z+i)^2) - (\operatorname{Log} z)\frac{d}{dz}((z+i)^2)}{((z+i)^2)^2} \\ &= \frac{\frac{1}{z}(z+i)^2 - (\operatorname{Log} z)(2(z+i))}{(z+i)^4} \\ &= \frac{(z+i)^2 - 2z(z+i)\operatorname{Log} z}{z(z+i)^4},\end{aligned}$$

and so the residue of $\dfrac{\operatorname{Log} z}{(z^2+1)^2}$ at $z = i$ is

$$\begin{aligned}
\operatorname*{Res}_{z=-1} \frac{\operatorname{Log} z}{(z+i)^2} &= \frac{\phi^{(2-1)}(i)}{(2-1)!} \\
&= \frac{\phi^{(1)}(i)}{1!} \\
&= \phi'(i) \\
&= \frac{(i+i)^2 - 2i(i+i)\operatorname{Log} i}{i(i+i)^4} \\
&= \frac{(2i)^2 - 2i(2i)(\ln(|i|) + i\operatorname{Arg}(i))}{i(2i)^4} \\
&= \frac{4i^2 - 4i^2(\ln(1) + i\frac{\pi}{2})}{i(16i^4)} \\
&= \frac{4(-1) - 4(-1)(0 + \frac{\pi}{2}i)}{i(16(1))} \\
&= \frac{-4 + 2\pi i}{16i} \\
&= \frac{-4 + 2\pi i}{16i}\frac{i}{i} \\
&= \frac{-4i + 2\pi i^2}{16i^2} \\
&= \frac{-4i + 2\pi(-1)}{16(-1)} \\
&= \frac{-4i - 2\pi}{-16} \\
&= \frac{-2\pi - 4i}{-16} \\
&= \frac{-2(\pi + 2i)}{-16} \\
&= \boxed{\frac{\pi + 2i}{8}}.
\end{aligned}$$

□

81.3 In each case, find the order m of the pole and the corresponding residue B at the singular point $z = 0$:

(b) $\dfrac{1}{z(e^z - 1)}$

Solution. For all z satisfying $|e^z| < 1$, which means for all $z = x + iy$ satisfying

$x = \operatorname{Re}(z) < 0$, we have

$$\begin{aligned}
e^z - 1 &= \sum_{k=0}^{\infty} \frac{z^k}{k!} - 1 \\
&= \left(1 + \sum_{k=1}^{\infty} \frac{z^k}{k!}\right) - 1 \\
&= \sum_{k=1}^{\infty} \frac{z^k}{k!} \\
&= \frac{z^1}{1!} + \sum_{k=2}^{\infty} \frac{z^k}{k!} \\
&= z + \sum_{k=2}^{\infty} \frac{z^k}{k!} \\
&= z + z \sum_{k=2}^{\infty} \frac{z^{k-1}}{k!} \\
&= z \left(1 + \sum_{k=2}^{\infty} \frac{z^{k-1}}{k!}\right),
\end{aligned}$$

which allows us to write

$$\begin{aligned}
\frac{1}{z(e^z-1)} &= \frac{1}{z(z(1+\sum_{k=2}^{\infty}\frac{z^{k-1}}{k!}))} \\
&= \frac{1}{z^2(1+\sum_{k=2}^{\infty}\frac{z^{k-1}}{k!})} \\
&= \frac{1}{z^2}\frac{1}{1-(-\sum_{k=2}^{\infty}\frac{z^{k-1}}{k!})} \\
&= \frac{1}{z^2}\sum_{n=0}^{\infty}\left(-\sum_{k=2}^{\infty}\frac{z^{k-1}}{k!}\right)^n \\
&= \frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n\left(\sum_{k=2}^{\infty}\frac{z^{k-1}}{k!}\right)^n \\
&= \frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z^{2-1}}{2!}+\frac{z^{3-1}}{3!}+\sum_{k=2}^{\infty}\frac{z^{k-1}}{k!}\right)^n \\
&= \frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}z+\frac{1}{3}z^2+\sum_{k=2}^{\infty}\frac{z^{k-1}}{k!}\right)^n \\
&= \frac{1}{z^2}\Bigg[(-1)^0\left(\frac{1}{2}z+\frac{1}{3}z^2+\sum_{k=2}^{\infty}\frac{z^{k-1}}{k!}\right)^0+(-1)^1\left(\frac{1}{2}z+\frac{1}{3}z^2+\sum_{k=2}^{\infty}\frac{z^{k-1}}{k!}\right)^1 \\
&\qquad +\sum_{n=2}^{\infty}(-1)^n\left(\frac{1}{2}z+\frac{1}{3}z^2+\sum_{k=2}^{\infty}\frac{z^{k-1}}{k!}\right)^n\Bigg] \\
&= \frac{1}{z^2}\Bigg[1\cdot 1+(-1)\left(\frac{1}{2}z+\frac{1}{3}z^2+\sum_{k=2}^{\infty}\frac{z^{k-1}}{k!}\right)^1 \\
&\qquad +\sum_{n=2}^{\infty}(-1)^n\left(\frac{1}{2}z+\frac{1}{3}z^2+\sum_{k=2}^{\infty}\frac{z^{k-1}}{k!}\right)^n\Bigg] \\
&= \frac{1}{z^2}\left[1-\frac{1}{2}z-\frac{1}{3}z^2-\sum_{k=2}^{\infty}\frac{z^{k-1}}{k!}+\sum_{n=2}^{\infty}(-1)^n\left(\frac{1}{2}z+\frac{1}{3}z^2+\sum_{k=2}^{\infty}\frac{z^{k-1}}{k!}\right)^n\right] \\
&= \frac{1}{z^2}-\frac{1}{2z}-\frac{1}{3}-\sum_{k=2}^{\infty}\frac{z^{k-3}}{k!}+\sum_{n=2}^{\infty}(-1)^n\frac{1}{z^2}\left(\frac{1}{2}z+\frac{1}{3}z^2+\sum_{k=2}^{\infty}\frac{z^{k-1}}{k!}\right)^n.
\end{aligned}$$

The final expression is a Laurent series expansion of $\frac{1}{z(e^z-1)}$ that holds for all z satisfying $\operatorname{Re}(z) < 0$. Since the Laurent series is unique, it also holds on the closure (interior and boundary) of the domain $\operatorname{Re}(z) < 0$, and this closure includes the point $z = 0$. Consequently, we see from our series expansion that $z = 0$ is a pole of order 2. Also, the residue at $z = 0$ is the coefficient of $\frac{1}{z}$ in our series expansion, which is

$$\operatorname*{Res}_{z=0}\frac{1}{z(e^z-1)} = -\frac{1}{2}.$$

□

81.4 Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz$$

taken clockwise around the circles:

(a) $|z-2| = 2$

Solution. The only singular point of the integrand inside the circle $|z-2| = 2$ is $z = 1$, which is a pole of order 1. For $z = 1$, we can write

$$\frac{3z^3+2}{(z-1)(z^2+9)} = \frac{\phi(z)}{z^2+9},$$

where we define

$$\phi(z) = \frac{3z^3+2}{z^2+9},$$

and so the residue of $\dfrac{3z^3+2}{(z-1)(z^2+9)}$ at $z = 1$ is

$$\begin{aligned}\operatorname*{Res}_{z=1} \frac{3z^3+2}{(z-1)(z^2+9)} &= \phi(1)\\ &= \frac{3(1)^3+2}{(1)^2+9}\\ &= \frac{5}{10}\\ &= \frac{1}{2}.\end{aligned}$$

By the theorem in Section 77, we obtain

$$\begin{aligned}\int_{|z-2|=2} \frac{3z^3+2}{(z-1)(z^2+9)}\,dz &= 2\pi i \operatorname*{Res}_{z=0} \frac{3z^3+2}{(z-1)(z^2+9)}\\ &= 2\pi i \frac{1}{2}\\ &= \boxed{\pi i}.\end{aligned}$$

□

(b) $|z| = 4$

Solution. The singularities of the integrand inside the circle $|z| = 3$ are $z = 1$ and $z = \pm 3i$, which are all poles of order 1. We already know

$$\operatorname*{Res}_{z=1} \frac{3z^3+2}{(z-1)(z^2+9)} = \frac{1}{2}$$

from part (a). For $z = 3i$, we can write

$$\begin{aligned}\frac{3z^3+2}{(z-1)(z^2+9)} &= \frac{3z^3+2}{(z-1)(z+3i)(z-3i)}\\ &= \frac{\phi(z)}{z-3i},\end{aligned}$$

where we define

$$\phi(z) = \frac{3z^3 + 2}{(z-1)(z+3i)},$$

and so the residue at $z = 3i$ is

$$\begin{aligned}
\operatorname*{Res}_{z=3i} \frac{3z^3+2}{(z-1)(z^2+9)} &= \phi(3i) \\
&= \frac{3(3i)^3 + 2}{(3i-1)(3i+3i)} \\
&= \frac{3(27i^3)+2}{(3i-1)6i} \\
&= \frac{81(-i)+2}{18i^2 - 6i} \\
&= \frac{2-81i}{18(-1)-6i} \\
&= \frac{2-81i}{-18-6i} \\
&= -\frac{2-81i}{18+6i} \\
&= -\frac{2-81i}{18+6i}\frac{18-6i}{18-6i} \\
&= -\frac{36-12i-1458i+486i^2}{324-108i+108i-36i^2} \\
&= -\frac{36-1470i+486(-1)}{324-36(-1)} \\
&= -\frac{36-1470i-486}{324+36} \\
&= -\frac{-450-1470i}{360} \\
&= -\frac{-30(15+49i)}{360} \\
&= \frac{15+49i}{12}.
\end{aligned}$$

For $z = 3i$, we can write

$$\begin{aligned}
\frac{3z^3+2}{(z-1)(z^2+9)} &= \frac{3z^3+2}{(z-1)(z+3i)(z-3i)} \\
&= \frac{\phi(z)}{z+3i},
\end{aligned}$$

where we define

$$\phi(z) = \frac{3z^3 + 2}{(z-1)(z-3i)},$$

and so the residue at $z = -3i$ is

$$\begin{aligned}
\operatorname*{Res}_{z=-3i} \frac{3z^3+2}{(z-1)(z^2+9)} &= \phi(-3i) \\
&= \frac{3(-3i)^3+2}{(-3i-1)(-3i-3i)} \\
&= \frac{3(-27i^3)+2}{-(3i+1)(-6i)} \\
&= \frac{-81(-i)+2}{18i^2+6i} \\
&= \frac{2+81i}{18(-1)+6i} \\
&= \frac{2+81i}{-18+6i} \\
&= -\frac{2+81i}{18-6i} \\
&= -\frac{2+81i}{18-6i}\,\frac{18+6i}{18+6i} \\
&= -\frac{36+12i+1458i+486i^2}{324-108i+108i-36i^2} \\
&= -\frac{36+1470i+486(-1)}{324-36(-1)} \\
&= -\frac{36+1470i-486}{324+36} \\
&= -\frac{-450+1470i}{360} \\
&= -\frac{-30(15-49i)}{360} \\
&= \frac{15-49i}{12}.
\end{aligned}$$

By the theorem in Section 77, we obtain

$$\begin{aligned}\int_{|z|=3} \frac{3z^3+2}{(z-1)(z^2+9)}\,dz &= 2\pi i \left(\operatorname*{Res}_{z=1} \frac{3z^3+2}{(z-1)(z^2+9)} + \operatorname*{Res}_{z=3i} \frac{3z^3+2}{(z-1)(z^2+9)} \right.\\ &\qquad \left. + \operatorname*{Res}_{z=-3i} \frac{3z^3+2}{(z-1)(z^2+9)}\right)\\ &= 2\pi i \left(\frac{1}{2} + \frac{15+49i}{12} + \frac{15-49i}{12}\right)\\ &= 2\pi i \left(\frac{1}{2} + \frac{(15+49i)+(15-49i)}{12}\right)\\ &= 2\pi i \left(\frac{1}{2} + \frac{30}{12}\right)\\ &= 2\pi i \left(\frac{1}{2} + \frac{5}{2}\right)\\ &= 2\pi i \left(\frac{6}{2}\right)\\ &= \boxed{6\pi i}.\end{aligned}$$

□

81.5 Find the value of the integral

$$\int_C \frac{1}{z^3(z+4)}\,dz$$

taken clockwise around the circles:

(a) $|z| = 2$

Solution. The only singular point of the integrand inside the circle $|z| = 2$ is $z = 0$, which is a pole of order 3. We can write $\dfrac{1}{z^3(z+4)}$ as a Laurent series about $z = 0$;

we have

$$
\begin{aligned}
\frac{1}{z^3(z+4)} &= \frac{1}{4z^3(\frac{z}{4}+1)} \\
&= \frac{1}{4z^3}\frac{1}{1+\frac{z}{4}} \\
&= \frac{1}{4z^3}\frac{1}{1-(-\frac{z}{4})} \\
&= \frac{1}{4z^3}\sum_{n=0}^{\infty}\left(-\frac{z}{4}\right)^n \\
&= \frac{1}{4}z^{-3}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}z^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}z^{n-3} \\
&= \frac{(-1)^0}{4^{0+1}}z^{0-3} + \frac{(-1)^1}{4^{1+1}}z^{1-3} + \frac{(-1)^2}{4^{2+1}}z^{2-3} + \sum_{n=3}^{\infty}\frac{(-1)^n}{4^{n+1}}z^{n-3} \\
&= \frac{1}{4}z^{-3} + \frac{1}{4^2}z^{-2} + \frac{1}{4^3}z^{-1} + \sum_{n=2}^{\infty}\frac{(-1)^n}{4^{n+1}}z^{n-3} \\
&= \frac{1}{4z^3} + \frac{1}{16z^2} + \frac{1}{64z} + \sum_{n=2}^{\infty}\frac{(-1)^n}{4^{n+1}}z^{n-3}.
\end{aligned}
$$

The residue at the singular point $z = 0$ is the coefficient of $\frac{1}{z}$, which is

$$\operatorname*{Res}_{z=0}\frac{1}{z^3(z+4)} = \frac{1}{64}.$$

By Cauchy's residue theorem, we conclude

$$
\begin{aligned}
\int_{|z|=3}\frac{1}{z^3(z+4)}\,dz &= 2\pi i\operatorname*{Res}_{z=0}\frac{1}{z^3(z+4)} \\
&= 2\pi i\left(\frac{1}{64}\right) \\
&= \boxed{\frac{\pi i}{32}}.
\end{aligned}
$$

□

(b) $|z+2| = 3$

Solution. The singularities of the integrand inside the circle $|z+2| = 3$ are $z = 0$ (a pole of order 3) and $z = -4$ (a pole of order 1). We already know

$$\operatorname*{Res}_{z=0}\frac{1}{z^3(z+4)} = \frac{1}{64}.$$

from part (a). For $z = -4$, we can write

$$\frac{1}{z^3(z+4)} = \frac{\phi(z)}{z+4},$$

where we define

$$\phi(z) = \frac{1}{z^3},$$

and so the residue at $z = -4$ is

$$\begin{aligned} \operatorname*{Res}_{z=-4} \frac{1}{z^3(z+4)} &= \phi(-4) \\ &= \frac{1}{(-4)^3} \\ &= \frac{1}{-64} \\ &= -\frac{1}{64}. \end{aligned}$$

By Cauchy's residue theorem, we conclude

$$\begin{aligned} \int_{|z|=3} \frac{1}{z^3(z+4)}\, dz &= 2\pi i \left(\operatorname*{Res}_{z=0} \frac{1}{z^3(z+4)} + \operatorname*{Res}_{z=-4} \frac{1}{z^3(z+4)} \right) \\ &= 2\pi i \left(\frac{1}{64} + \left(-\frac{1}{64} \right) \right) \\ &= 2\pi i(0) \\ &= \boxed{0}. \end{aligned}$$

□

81.7 Use the theorem in Section 77, involving a single residue, to evaluate the integral of $f(z)$ around the positively oriented circle $|z| = 3$ when

(a) $f(z) = \dfrac{(3z+2)^2}{z(z-1)(2z+5)}$

Solution. By setting $f(z) = \dfrac{(3z+2)^2}{z(z-1)(2z+5)}$, we have

$$\begin{aligned} f\left(\frac{1}{z}\right) &= \frac{(3(\frac{1}{z})+2)^2}{\frac{1}{z}(\frac{1}{z}-1)(2(\frac{1}{z})+5)} \\ &= \frac{(\frac{3}{z}+2)^2}{\frac{1}{z}(\frac{1}{z}-1)(\frac{2}{z}+5)} \\ &= \frac{(\frac{3}{z}+2)^2}{\frac{1}{z}(\frac{1}{z}-1)(\frac{2}{z}+5)} \frac{z^3}{z^3} \\ &= \frac{zz^2(\frac{3}{z}+2)^2}{z\frac{1}{z}z(\frac{1}{z}-1)z(\frac{2}{z}+5)} \\ &= \frac{z(3+2z)^2}{(1-z)(2+5z)}, \end{aligned}$$

and so we have

$$
\begin{aligned}
\frac{1}{z^2} f\left(\frac{1}{z}\right) &= \frac{1}{z^2} \frac{z(3+2z)^2}{(1-z)(2+5z)} \\
&= \frac{1}{z} \frac{(3+2z)^2}{(1-z)(2+5z)} \\
&= \frac{1}{z} \left(\frac{(3+2(0))^2}{(1-0)(2+5(0))} + \sum_{n=1}^{\infty} a_n z^n \right) \\
&= \frac{1}{z} \left(\frac{9}{2} + \sum_{n=1}^{\infty} a_n z^n \right) \\
&= \frac{9}{2z} + \sum_{n=1}^{\infty} a_n z^{n-1},
\end{aligned}
$$

where a_n for $n = 1, 2, 3, \ldots$ are suitable complex coefficients for the Laurent series expansion about $z = 0$ of $\frac{1}{z^2} f\left(\frac{1}{z}\right)$. The residue at the singular point $z = 0$ is the coefficient of $\frac{1}{z}$, which is

$$
\operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right] = \frac{9}{2}.
$$

By the theorem in Section 77, we obtain

$$
\begin{aligned}
\int_{|z|=2} \frac{z^5}{1-z^3}\, dz &= 2\pi i \operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right] \\
&= 2\pi i \left(\frac{9}{2}\right) \\
&= \boxed{9\pi i}.
\end{aligned}
$$

□