MATH 165B discussion Ryan Ta
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Solutions to suggested homework problems from
Complex Variables and Applications, Ninth Edition by James Brown and Ruel Churchill
Homework 4: Section 83, Exercises 1, 2, 4(a), 5(a), 8, 9(a)(b)

83.1. Show that the point $z = 0$ is a simple pole of the function

$$f(z) = \csc z = \frac{1}{\sin z}$$

and that the residue there is unity by appealing to Theorem 2 in Section 83. (Compare with Exercise 3, Section 73, where this result is evident from a Laurent series.)

Solution. We can write

$$\begin{aligned} f(z) &= \csc z \\ &= \frac{1}{\sin z} \\ &= \frac{p(z)}{q(z)}, \end{aligned}$$

where we define

$$\begin{aligned} p(z) &= 1, \\ q(z) &= \sin z, \end{aligned}$$

both of which are analytic at $z_0 = 0$. Also, the first derivative of q is

$$\begin{aligned} q'(z) &= \frac{d}{dz}(\sin z) \\ &= \cos z. \end{aligned}$$

At the point $z_0 = 0$, we have

$$\begin{aligned} p(z_0) &= p(0) \\ &= 1 \\ &\neq 0 \end{aligned}$$

and

$$\begin{aligned} q(z_0) &= q(0) \\ &= \sin 0 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} q'(z_0) &= q'(0) \\ &= \cos 0 \\ &= 1 \\ &\neq 0. \end{aligned}$$

By Theorem 2 of Section 83, $z_0 = 0$ is a simple pole with residue

$$\begin{aligned}\operatorname*{Res}_{z=0} \frac{p(z)}{q(z)} &= \frac{p(0)}{q'(0)} \\ &= \frac{1}{1} \\ &= \boxed{1},\end{aligned}$$

which is unity. □

83.2. Use conditions (1) in Section 82 to show that the function

$$q(z) = 1 - \cos z$$

has a zero of order $m = 2$ at the point $z_0 = 0$.

Solution. Recall the Maclaurin series of $\cos z$, which is

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

We have

$$\begin{aligned} q(z) &= 1 - \cos z \\ &= 1 - \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \\ &= 1 - \left((-1)^0 \frac{z^{2(0)}}{(2(0))!} + \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \right) \\ &= 1 - \left(1 + \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \right) \\ &= 1 - 1 - \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \\ &= - \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \\ &= - \sum_{n=1}^{\infty} (-1)^n \frac{z^2 z^{2n-2}}{(2n)!} \\ &= -(z-0)^2 \sum_{n=1}^{\infty} (-1)^n \frac{z^{2(n-1)}}{(2n)!}, \end{aligned}$$

and so $q(z)$ has a zero of order $m = 2$ at the point $z_0 = 0$. □

83.4. Show that

(a) $\operatorname*{Res}_{z=z_n}(z \sec z) = (-1)^{n+1} z_n$, where $z_n = \dfrac{\pi}{2} + n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$.

Proof. We can write

$$\begin{aligned} z \sec z &= \frac{z}{\cos z} \\ &= \frac{p(z)}{q(z)}, \end{aligned}$$

where we define

$$\begin{aligned} p(z) &= z, \\ q(z) &= \cos z, \end{aligned}$$

both of which are analytic at $z_0 = 0$. Also, the first derivative of q is

$$\begin{aligned} q'(z) &= \frac{d}{dz}(\cos z) \\ &= -\sin z. \end{aligned}$$

At the points $z_n = \frac{\pi}{2} + n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$, we have

$$\begin{aligned} p(z_n) &= z_n \\ &= \frac{\pi}{2} + n\pi \\ &\neq 0 \end{aligned}$$

and

$$\begin{aligned} q(z_n) &= \cos(z_n) \\ &= \cos\left(\frac{\pi}{2} + n\pi\right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} q'(z_n) &= -\sin(z_n) \\ &= -\sin\left(\frac{\pi}{2} + n\pi\right) \\ &= -(-1)^n \\ &= (-1)^{n+1} \\ &\neq 0. \end{aligned}$$

By Theorem 2 of Section 83, we conclude

$$\begin{aligned} \operatorname*{Res}_{z=z_n}(z \sec z) &= \operatorname*{Res}_{z=z_n} \frac{p(z)}{q(z)} \\ &= \frac{p(z_n)}{q'(z_n)} \\ &= \frac{z_n}{(-1)^{n+1}} \\ &= (-1)^{-(n+1)} z_n \\ &= (-1)^{n+1} z_n, \end{aligned}$$

as desired. □

83.5. Let C denote the positively oriented circle $|z| = 2$ and evaluate the integral

(a) $\displaystyle\int_C \tan z \, dz$.

Solution. We can write

$$\begin{aligned} \tan z &= \frac{\sin z}{\cos z} \\ &= \frac{p(z)}{q(z)}, \end{aligned}$$

where we define

$$\begin{aligned} p(z) &= \sin z, \\ q(z) &= \cos z, \end{aligned}$$

both of which are analytic at $z_0 = 0$. Also, the first derivative of q is

$$\begin{aligned} q'(z) &= \frac{d}{dz}(\cos z) \\ &= -\sin z. \end{aligned}$$

At the points $z_n = \frac{\pi}{2} + n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$, which are the singularities of $\tan z$, we have

$$\begin{aligned} p(z_n) &= \sin(z_n) \\ &= \sin\left(\frac{\pi}{2} + n\pi\right) \\ &= (-1)^n \\ &\neq 0 \end{aligned}$$

and

$$\begin{aligned} q(z_n) &= \cos(z_n) \\ &= \cos\left(\frac{\pi}{2} + n\pi\right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} q'(z_n) &= -\sin(z_n) \\ &= -\sin\left(\frac{\pi}{2} + n\pi\right) \\ &= -(-1)^n \\ &= (-1)^{n+1} \\ &\neq 0. \end{aligned}$$

By Theorem 2 of Section 83, we conclude

$$\begin{aligned}\operatorname*{Res}_{z=z_n}(\tan z) &= \operatorname*{Res}_{z=z_n}\frac{p(z)}{q(z)} \\ &= \frac{p(z_n)}{q'(z_n)} \\ &= \frac{(-1)^n}{(-1)^{n+1}} \\ &= (-1)^{n-(n+1)} \\ &= (-1)^{-1} \\ &= -1.\end{aligned}$$

But since the only two singularities of $\tan z$ inside the circle $|z| = 2$ are $z_0 = \frac{\pi}{2}$ and $z_{-1} = -\frac{\pi}{2}$, we have in particular

$$\operatorname*{Res}_{z=\frac{\pi}{2}}(\tan z) = -1,$$
$$\operatorname*{Res}_{z=-\frac{\pi}{2}}(\tan z) = -1.$$

By the theorem in Section 77 (residue theorem), we obtain

$$\begin{aligned}\int_{|z|=2} \tan z\, dz &= 2\pi i\left(\operatorname*{Res}_{z=\frac{\pi}{2}}(\tan z) + \operatorname*{Res}_{z=-\frac{\pi}{2}}(\tan z)\right) \\ &= 2\pi i((-1) + (-1)) \\ &= 2\pi i(-2) \\ &= \boxed{-4\pi i}.\end{aligned}$$

□

83.8. Consider the function

$$f(z) = \frac{1}{[q(z)]^2},$$

where q is analytic at z_0, $q(z_0) = 0$, and $q'(z_0) \neq 0$. Show that z_0 is a pole of order $m = 2$ of the function f, with residue

$$B_0 = -\frac{q''(z_0)}{[q'(z_0)]^3}.$$

Suggestion: Note that z_0 is a zero of order $m = 1$ of the function q, so that

$$q(z) = (z - z_0)g(z),$$

where $g(z)$ is analytic and nonzero at z_0. Then write

$$f(z) = \frac{\phi(z)}{(z - z_0)^2},$$

where $\phi(z) = \frac{1}{[g(z)]^2}$. The desired form of the residue $B_0 = \phi'(z_0)$ can be obtained by showing that

$$q'(z_0) = g(z_0)$$

and

$$q''(z_0) = 2g'(z_0).$$

Solution. Following the given suggestion, note that z_0 is a zero of order $m = 1$ of the function q, so that we have

$$q(z) = (z - z_0)g(z),$$

where g is analytic and nonzero at z_0. Then we have

$$\begin{aligned} f(z) &= \frac{1}{[q(z)]^2} \\ &= \frac{1}{[(z - z_0)g(z)]^2} \\ &= \frac{1}{(z - z_0)^2[g(z)]^2} \\ &= \frac{\phi(z)}{(z - z_0)^2}, \end{aligned}$$

where we define $\phi(z) = \dfrac{1}{[g(z)]^2}$. Since g is analytic and nonzero, it follows that ϕ is analytic and nonzero as well, which implies that $z = z_0$ is a pole of order $m = 2$ for f. The first derivative of ϕ is

$$\begin{aligned} \phi'(z) &= \frac{d}{dz}\left(\frac{1}{[g(z)]^2}\right) \\ &= \frac{d}{dz}[g(z)]^{-2} \\ &= -2[g(z)]^{-3}g'(z) \\ &= -\frac{2g'(z)}{[g(z)]^3}. \end{aligned}$$

Since we have established that $z = z_0$ is a pole of order $m = 2$, the theorem in Section 80 asserts that the residue at $z = z_0$ is

$$\begin{aligned} \operatorname*{Res}_{z=z_0} f(z) &= \frac{\phi^{(2-1)}(z_0)}{(2-1)!} \\ &= \frac{\phi^{(1)}(z_0)}{1!} \\ &= \phi'(z_0) \\ &= -\frac{2g'(z_0)}{[g(z_0)]^3}. \end{aligned}$$

Furthermore, the first derivative of q is

$$\begin{aligned} q'(z) &= \frac{d}{dz}(q(z)) \\ &= \frac{d}{dz}((z - z_0)g(z)) \\ &= \frac{d}{dz}(z - z_0)g(z) + (z - z_0)\frac{d}{dz}(g(z)) \\ &= 1g(z) + (z - z_0)g'(z) \\ &= g(z) + (z - z_0)g'(z), \end{aligned}$$

and the second derivative of q is

$$\begin{aligned}
q''(z) &= \frac{d}{dz}(q'(z)) \\
&= \frac{d}{dz}(g'(z) + (z - z_0)g'(z)) \\
&= \frac{d}{dz}(g'(z)) + \frac{d}{dz}((z - z_0)g'(z)) \\
&= g'(z) + (\frac{d}{dz}(z - z_0)g'(z) + (z - z_0)\frac{d}{dz}(g'(z))) \\
&= g'(z) + (1g'(z) + (z - z_0)g''(z)) \\
&= 2g'(z) + (z - z_0)g''(z).
\end{aligned}$$

At the point $z = z_0$, we have

$$\begin{aligned}
q'(z_0) &= g(z_0) + (z_0 - z_0)g''(z_0) \\
&= g(z_0) + 0g''(z_0) \\
&= g(z_0)
\end{aligned}$$

and

$$\begin{aligned}
q''(z_0) &= 2g'(z_0) + (z_0 - z_0)g''(z_0) \\
&= 2g'(z_0) + 0g''(z_0) \\
&= 2g'(z_0).
\end{aligned}$$

Finally, we conclude

$$\begin{aligned}
\operatorname*{Res}_{z=z_0} f(z) &= -\frac{2g'(z_0)}{[g(z_0)]^3} \\
&= -\frac{q''(z_0)}{[q'(z_0)]^3},
\end{aligned}$$

as desired. □

83.9. Use the result in Exercise **8** to find the residue at $z = 0$ of the function

(a) $f(z) = \csc^2 z$.

Solution. We can write

$$\begin{aligned}
\csc^2 z &= \frac{1}{\sin^2 z} \\
&= \frac{1}{[q(z)]^2},
\end{aligned}$$

where we define

$$q(z) = \sin z,$$

which is analytic at $z = 0$.

$$\begin{aligned}
q'(z) &= \frac{d}{dz}(\cos z) \\
&= -\sin z,
\end{aligned}$$

and the second derivative of q is

$$\begin{aligned} q''(z) &= \frac{d}{dz}(-\sin z) \\ &= -\cos z. \end{aligned}$$

Also, we have

$$\begin{aligned} q(0) &= \sin 0 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} q'(0) &= \cos 0 \\ &= 1 \\ &\neq 0. \end{aligned}$$

By Exercise 8, the residue at $z = 0$ of $\csc^2 z$ is

$$\begin{aligned} \operatorname*{Res}_{z=0}(\csc^2 z) &= -\frac{q''(0)}{[q'(0)]^3} \\ &= -\frac{-\sin 0}{(\cos 0)^3} \\ &= -\frac{-0}{1^3} \\ &= \boxed{0}. \end{aligned}$$

□

(b) $f(z) = \dfrac{1}{(z + z^2)^2}$.

Solution. We can write

$$\frac{1}{(z + z^2)^2} = \frac{1}{[q(z)]^2},$$

where we define

$$q(z) = z + z^2,$$

which is analytic at $z = 0$. The first derivative of q is

$$\begin{aligned} q'(z) &= \frac{d}{dz}(z + z^2) \\ &= 1 + 2z, \end{aligned}$$

and the second derivative of q is

$$\begin{aligned} q''(z) &= \frac{d}{dz}(1 + 2z) \\ &= 2. \end{aligned}$$

Also, we have

$$\begin{aligned} q(0) &= 0 + 0^2 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} q'(0) &= 1 + 2(0) \\ &= 1 \\ &\neq 0. \end{aligned}$$

By Exercise 8, the residue at $z = 0$ of $\dfrac{1}{(z+z^2)^2}$ is

$$\begin{aligned} \operatorname*{Res}_{z=0} \frac{1}{(z+z^2)^2} &= -\frac{q''(0)}{[q'(0)]^3} \\ &= -\frac{2}{(1+2(0))^3} \\ &= -\frac{2}{1^3} \\ &= \boxed{-2}. \end{aligned}$$

□