MATH 165B discussion
University of California, Riverside

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Solutions to suggested homework problems from
Complex Variables and Applications, Ninth Edition by James Brown and Ruel Churchill
Homework 5: Section 86, Exercises 1, 2, 3, 4, 5, 7, 8

Use residues to derive the integration formulas in Exercises 1 through 5.

86.1. $\displaystyle\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}.$

Solution. The isolated singularities of

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

are $z = \pm i$, each of which are poles of order 1. But only $z = i$ lies within the upper semicircle C_R with a sufficiently large radius $R > 0$. The residue of $\frac{1}{z^2+1}$ at $z = i$ is

$$\begin{aligned}
\operatorname*{Res}_{z=i} \frac{1}{z^2+1} &= \left.\frac{1}{z+i}\right|_{z=i} \\
&= \frac{1}{i+1} \\
&= \frac{1}{2i} \\
&= \frac{1}{2i}\frac{i}{i} \\
&= \frac{i}{2i^2} \\
&= \frac{i}{2(-1)} \\
&= -\frac{i}{2}.
\end{aligned}$$

By the residue theorem from Section 77 applied to the region bounded by $[-R, R]$ and C_R, we have

$$\begin{aligned}
\int_{-R}^{R} \frac{dx}{x^2+1} &= 2\pi i \operatorname*{Res}_{z=i} \frac{1}{z^2+1} - \int_{C_R} \frac{dz}{z^2+1} \\
&= 2\pi i\left(-\frac{i}{2}\right) - \int_{C_R} \frac{dz}{z^2+1} \\
&= -\pi i^2 - \int_{C_R} \frac{dz}{z^2+1} \\
&= -\pi(-1) - \int_{C_R} \frac{dz}{z^2+1} \\
&= \pi - \int_{C_R} \frac{dz}{z^2+1}.
\end{aligned}$$

Next, if we substitute $z = Re^{i\theta}$ and $dz = iRe^{i\theta}\,d\theta$ for all $0 < \theta < \pi$, we obtain

$$
\begin{aligned}
\left|\int_{C_R} \frac{dz}{z^2+1}\right| &= \left|\int_{C_R} \frac{iRe^{i\theta}\,d\theta}{(Re^{i\theta})^2+1}\right| \\
&= \left|\int_0^\pi \frac{iRe^{i\theta}\,d\theta}{R^2e^{2i\theta}+1}\right| \\
&\le \int_0^\pi \frac{|iRe^{i\theta}|}{|R^2e^{2i\theta}+1|}\,d\theta \\
&\le \int_0^\pi \frac{|iRe^{i\theta}|}{||R^2e^{2i\theta}|-|1||}\,d\theta \\
&= \int_0^\pi \frac{R}{|R^2-1|}\,d\theta \\
&= \frac{R}{R^2-1}\int_0^\pi 1\,d\theta \\
&= \frac{R}{R^2-1}\theta\Big|_0^\pi \\
&= \frac{R}{R^2-1}(\pi-0) \\
&= \frac{\pi R}{R^2-1},
\end{aligned}
$$

which implies

$$
\begin{aligned}
0 &\le \left|\lim_{R\to\infty}\int_{C_R} \frac{dz}{z^2+1}\right| \\
&= \lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{z^2+1}\right| \\
&\le \lim_{R\to\infty}\frac{\pi R}{R^2-1} \\
&= \lim_{R\to\infty}\frac{\pi R}{R^2-1}\frac{\frac{1}{R}}{\frac{1}{R}} \\
&= \lim_{R\to\infty}\frac{\pi}{R-\frac{1}{R}} \\
&= \frac{\pi}{\infty-0} \\
&= 0,
\end{aligned}
$$

from which we conclude

$$
\lim_{R\to\infty}\int_{C_R} \frac{dz}{z^2+1} = 0.
$$

Finally, since $\dfrac{1}{x^2+1}$ is an even function, we conclude

$$\begin{aligned}\int_0^\infty \frac{dx}{x^2+1} &= \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{x^2+1}\\ &= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{x^2+1}\\ &= \frac{1}{2}\lim_{R\to\infty}\left(\pi - \int_{C_R}\frac{dz}{z^2+1}\right)\\ &= \frac{1}{2}\left(\lim_{R\to\infty}\pi - \lim_{R\to\infty}\int_{C_R}\frac{dz}{z^2+1}\right)\\ &= \frac{1}{2}(\pi - 0)\\ &= \boxed{\frac{\pi}{2}}.\end{aligned}$$

□

86.2. $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$

Solution. The isolated singularities of

$$\begin{aligned}\frac{1}{(z^2+1)^2} &= \frac{1}{((z+i)(z-i))^2}\\ &= \frac{1}{(z+i)^2(z-i)^2}\end{aligned}$$

are $z = \pm i$, each of which are poles of order 2. But only $z = i$ lies within the upper semicircle C_R with a sufficiently large radius $R > 0$. The residue of $\dfrac{1}{(z^2+1)^2}$ at $z = i$ is

$$\begin{aligned}
\operatorname*{Res}_{z=i} \frac{1}{(z^2+1)^2} &= \frac{1}{(2-1)!} \left(\frac{1}{(z+i)^2} \right)^{(2-1)} \Bigg|_{z=i} \\
&= \left(\frac{1}{(z+i)^2} \right)' \Bigg|_{z=i} \\
&= -\frac{2}{(z+i)^3} \Bigg|_{z=i} \\
&= -\frac{2}{(i+i)^3} \\
&= -\frac{2}{(2i)^3} \\
&= -\frac{2}{8i^3} \\
&= -\frac{2}{8(-i)} \\
&= \frac{1}{4i} \\
&= \frac{1}{4i}\frac{i}{i} \\
&= \frac{i}{4i^2} \\
&= \frac{i}{4(-1)} \\
&= -\frac{i}{4}.
\end{aligned}$$

By the residue theorem from Section 77 applied to the region bounded by $[-R, R]$ and C_R, we have

$$\begin{aligned}
\int_{-R}^{R} \frac{dx}{(x^2+1)^2} &= 2\pi i \operatorname*{Res}_{z=i} \frac{1}{(z^2+1)^2} - \int_{C_R} \frac{dz}{(z^2+1)^2} \\
&= 2\pi i \left(-\frac{i}{4} \right) - \int_{C_R} \frac{dz}{(z^2+1)^2} \\
&= -\frac{\pi i^2}{2} - \int_{C_R} \frac{dz}{(z^2+1)^2} \\
&= -\frac{\pi(-1)}{2} - \int_{C_R} \frac{dz}{(z^2+1)^2} \\
&= \frac{\pi}{2} - \int_{C_R} \frac{dz}{(z^2+1)^2}.
\end{aligned}$$

Next, if we substitute $z = Re^{i\theta}$ and $dz = iRe^{i\theta}\,d\theta$ for all $0 < \theta < \pi$, we obtain

$$
\begin{aligned}
\left|\int_{C_R} \frac{dz}{(z^2+1)^2}\right| &= \left|\int_0^{\pi} \frac{iRe^{i\theta}\,d\theta}{((Re^{i\theta})^2+1)^2}\right| \\
&= \left|\int_0^{\pi} \frac{iRe^{i\theta}\,d\theta}{(Re^{i\theta})^4+2(Re^{i\theta})^2+1}\right| \\
&= \left|\int_0^{\pi} \frac{iRe^{i\theta}\,d\theta}{R^4e^{4i\theta}+2R^2e^{2i\theta}+1}\right| \\
&\le \int_0^{\pi} \frac{|iRe^{i\theta}|}{|R^4e^{4i\theta}+2R^2e^{2i\theta}+1|}\,d\theta \\
&\le \int_0^{\pi} \frac{|iRe^{i\theta}|}{||R^4e^{4i\theta}|-|2R^2e^{2i\theta}|-|1||}\,d\theta \\
&= \int_0^{\pi} \frac{R}{|R^4-2R^2-1|}\,d\theta \\
&= \frac{R}{R^4-2R^2-1}\int_0^{\pi} 1\,d\theta \\
&= \frac{R}{R^4-2R^2-1}\theta\Big|_0^{\pi} \\
&= \frac{R}{R^4-2R^2-1}(\pi-0) \\
&= \frac{\pi R}{R^4-2R^2-1},
\end{aligned}
$$

which implies

$$
\begin{aligned}
0 &\le \left|\lim_{R\to\infty}\int_{C_R} \frac{dz}{z^2+1}\right| \\
&= \lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{z^2+1}\right| \\
&\le \lim_{R\to\infty} \frac{\pi R}{R^4-2R^2-1} \\
&= \lim_{R\to\infty} \frac{\pi R}{R^4-2R^2-1}\frac{\frac{1}{R}}{\frac{1}{R}} \\
&= \lim_{R\to\infty} \frac{\pi}{R^3-2R-\frac{1}{R}} \\
&= \lim_{R\to\infty} \frac{\pi}{R(R^2-2-\frac{1}{R^2})} \\
&= \frac{\pi}{\infty(\infty-2-0)} \\
&= 0,
\end{aligned}
$$

from which we conclude

$$
\lim_{R\to\infty}\int_{C_R} \frac{dz}{(z^2+1)^2} = 0.
$$

Finally, since $\frac{1}{(x^2+1)^2}$ is an even function, we conclude

$$\begin{aligned}\int_0^{\infty} \frac{dx}{(x^2+1)^2} &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} \\ &= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{(x^2+1)^2} \\ &= \frac{1}{2}\lim_{R\to\infty}\left(\frac{\pi}{2} - \int_{C_R} \frac{dz}{(z^2+1)^2}\right) \\ &= \frac{1}{2}\left(\lim_{R\to\infty}\frac{\pi}{2} - \lim_{R\to\infty}\int_{C_R} \frac{dz}{(z^2+1)^2}\right) \\ &= \frac{1}{2}\left(\frac{\pi}{2} - 0\right) \\ &= \boxed{\frac{\pi}{4}}.\end{aligned}$$

□

86.3. $\displaystyle\int_0^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$.

Solution. The isolated singularities of

$$\begin{aligned}\frac{1}{z^4+1} &= \frac{1}{(z^2+i)(z^2-i)} \\ &= \frac{1}{(z-\frac{-1+i}{\sqrt{2}})(z-\frac{-1-i}{\sqrt{2}})(z-\frac{1+i}{\sqrt{2}})(z-\frac{1-i}{\sqrt{2}})}\end{aligned}$$

are $z = \frac{-1\pm i}{\sqrt{2}}$ and $z = \frac{1\pm i}{\sqrt{2}}$, each of which are poles of order 1. But only $z = \frac{-1+i}{\sqrt{2}}$ and $z = \frac{1+i}{\sqrt{2}}$ lie within the upper semicircle C_R with a sufficiently large radius $R > 0$.

The residue of $\frac{1}{z^4+1}$ at $z = \frac{-1+i}{\sqrt{2}}$ is

$$\begin{aligned}
\operatorname*{Res}_{z=\frac{-1+i}{\sqrt{2}}} \frac{1}{z^4+1} &= \left.\frac{1}{(z - \frac{-1-i}{\sqrt{2}})(z - \frac{1+i}{\sqrt{2}})(z - \frac{1-i}{\sqrt{2}})}\right|_{z=\frac{-1+i}{\sqrt{2}}} \\
&= \frac{1}{(\frac{-1+i}{\sqrt{2}} - \frac{-1-i}{\sqrt{2}})(\frac{-1+i}{\sqrt{2}} - \frac{1+i}{\sqrt{2}})(\frac{-1+i}{\sqrt{2}} - \frac{1-i}{\sqrt{2}})} \\
&= \frac{1}{(\frac{2i}{\sqrt{2}})(\frac{-2}{\sqrt{2}})(\frac{-2+2i}{\sqrt{2}})} \\
&= \frac{1}{(\sqrt{2}i)(-\sqrt{2})(\sqrt{2}(-1+i))} \\
&= -\frac{1}{2\sqrt{2}i(-1+i)} \\
&= -\frac{1}{2\sqrt{2}(-i+i^2)} \\
&= -\frac{1}{2\sqrt{2}(-i+(-1))} \\
&= \frac{1}{2\sqrt{2}(1+i)} \frac{1-i}{1-i} \\
&= \frac{1-i}{2\sqrt{2}(1-i^2)} \\
&= \frac{1-i}{2\sqrt{2}(1-(-1))} \\
&= \frac{1-i}{2\sqrt{2}(2)} \\
&= \frac{1-i}{4\sqrt{2}},
\end{aligned}$$

and the residue of $\frac{1}{z^4+1}$ at $z = \frac{1+i}{\sqrt{2}}$ is

$$\begin{aligned}
\operatorname*{Res}_{z=\frac{1+i}{\sqrt{2}}} \frac{1}{z^4+1} &= \left.\frac{1}{(z-\frac{-1+i}{\sqrt{2}})(z-\frac{-1-i}{\sqrt{2}})(z-\frac{1-i}{\sqrt{2}})}\right|_{z=\frac{1+i}{\sqrt{2}}} \\
&= \frac{1}{(\frac{1+i}{\sqrt{2}}-\frac{-1+i}{\sqrt{2}})(\frac{1+i}{\sqrt{2}}-\frac{-1-i}{\sqrt{2}})(\frac{1+i}{\sqrt{2}}-\frac{1-i}{\sqrt{2}})} \\
&= \frac{1}{(\frac{2}{\sqrt{2}})(\frac{2+2i}{\sqrt{2}})(\frac{2i}{\sqrt{2}})} \\
&= \frac{1}{(\sqrt{2})(\sqrt{2}(1+i))(\sqrt{2}i)} \\
&= \frac{1}{2\sqrt{2}i(1+i)} \\
&= \frac{1}{2\sqrt{2}(i+i^2)} \\
&= \frac{1}{2\sqrt{2}(i+(-1))} \\
&= \frac{1}{2\sqrt{2}(-1+i)}\frac{-1-i}{-1-i} \\
&= \frac{-1-i}{2\sqrt{2}(1-i^2)} \\
&= \frac{-1-i}{2\sqrt{2}(1-(-1))} \\
&= \frac{-1-i}{2\sqrt{2}(2)} \\
&= \frac{-1-i}{4\sqrt{2}}.
\end{aligned}$$

By the residue theorem from Section 77 applied to the region bounded by $[-R, R]$ and C_R, we have

$$\begin{aligned}
\int_{-R}^{R} \frac{dx}{x^4+1} &= 2\pi i\left(\operatorname*{Res}_{z=\frac{-1+i}{\sqrt{2}}} \frac{1}{z^4+1} + \operatorname*{Res}_{z=\frac{1+i}{\sqrt{2}}} \frac{1}{z^4+1}\right) - \int_{C_R} \frac{dz}{z^4+1} \\
&= 2\pi i\left(\frac{1-i}{4\sqrt{2}} + \frac{-1-i}{4\sqrt{2}}\right) - \int_{C_R} \frac{dz}{z^4+1} \\
&= 2\pi i\left(\frac{-2i}{4\sqrt{2}}\right) - \int_{C_R} \frac{dz}{z^4+1} \\
&= -\frac{\pi i^2}{\sqrt{2}} - \int_{C_R} \frac{dz}{z^4+1} \\
&= -\frac{\pi(-1)}{\sqrt{2}} - \int_{C_R} \frac{dz}{z^4+1} \\
&= \frac{\pi}{\sqrt{2}} - \int_{C_R} \frac{dz}{z^4+1}.
\end{aligned}$$

Next, if we substitute $z = Re^{i\theta}$ and $dz = iRe^{i\theta}\,d\theta$ for all $0 < \theta < \pi$, we obtain

$$\begin{aligned}
\left|\int_{C_R} \frac{dz}{z^4+1}\right| &= \left|\int_0^\pi \frac{iRe^{i\theta}\,d\theta}{(Re^{i\theta})^4+1}\right| \\
&= \left|\int_0^\pi \frac{iRe^{i\theta}\,d\theta}{R^4e^{4i\theta}+1}\right| \\
&\leq \int_0^\pi \frac{|iRe^{i\theta}|}{|R^4e^{4i\theta}+1|}\,d\theta \\
&\leq \int_0^\pi \frac{|iRe^{i\theta}|}{||R^4e^{4i\theta}|-|1||}\,d\theta \\
&\leq \int_0^\pi \frac{R}{|R^4-1|}\,d\theta \\
&= \frac{R}{R^4-1}\int_0^\pi 1\,d\theta \\
&= \frac{R}{R^4-1}\theta|_0^\pi \\
&= \frac{R}{R^4-1}(\pi-0) \\
&= \frac{\pi R}{R^4-1},
\end{aligned}$$

which implies

$$\begin{aligned}
0 &\leq \left|\lim_{R\to\infty}\int_{C_R} \frac{dz}{z^4+1}\right| \\
&= \lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{z^4+1}\right| \\
&\leq \lim_{R\to\infty} \frac{\pi R}{R^4-1} \\
&= \lim_{R\to\infty} \frac{\pi R}{R^4-1}\frac{\frac{1}{R}}{\frac{1}{R}} \\
&= \lim_{R\to\infty} \frac{\pi}{R^3-\frac{1}{R}} \\
&= \frac{\pi}{\infty(\infty-0)} \\
&= 0,
\end{aligned}$$

from which we conclude

$$\lim_{R\to\infty}\int_{C_R} \frac{dz}{z^4+1} = 0.$$

Finally, since $\dfrac{1}{x^4+1}$ is an even function, we conclude

$$\begin{aligned}
\int_0^\infty \frac{dx}{x^4+1} &= \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{x^4+1} \\
&= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{x^4+1} \\
&= \frac{1}{2}\lim_{R\to\infty}\left(\frac{\pi}{\sqrt{2}} - \int_{C_R}\frac{dz}{z^4+1}\right) \\
&= \frac{1}{2}\left(\lim_{R\to\infty}\frac{\pi}{\sqrt{2}} - \lim_{R\to\infty}\int_{C_R}\frac{dz}{z^4+1}\right) \\
&= \frac{1}{2}\left(\frac{\pi}{\sqrt{2}} - 0\right) \\
&= \boxed{\frac{\pi}{2\sqrt{2}}}.
\end{aligned}$$

□

86.4. $\displaystyle\int_0^\infty \frac{x^2\,dx}{x^6+1} = \frac{\pi}{6}.$

Solution. Although one can compute this exercise by applying the residue theorem directly, as we have done in the previous exercises, it is possible to use only methods from first-year calculus where applicable while directly invoking Exercise 86.1. With that in mind, if we substitute $u = x^3$ and $du = 3x^2\,dx$, we obtain

$$\begin{aligned}
\int_0^\infty \frac{x^2\,dx}{x^6+1} &= \int_0^\infty \frac{x^2\,dx}{(x^3)^2+1} \\
&= \int_{0^3}^{\infty^3} \frac{1}{u^2+1}\frac{du}{3} \\
&= \frac{1}{3}\int_0^\infty \frac{du}{u^2+1} \\
&= \frac{1}{3}\left(\frac{\pi}{2}\right) \quad \text{by Exercise 86.1} \\
&= \boxed{\frac{\pi}{6}}.
\end{aligned}$$

□

86.5. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}.$

Solution. Although one can compute this exercise by applying the residue theorem directly, as we have done in the previous exercises, it is possible to use only methods from first-year calculus where applicable while directly invoking Exercise 86.1. With that in mind, if we employ the method of decomposition by partial fractions, we can write our integrand as

$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4},$$

which implies

$$\begin{aligned} x^2 &= (x^2+1)(x^2+4)\frac{x^2}{(x^2+1)(x^2+4)} \\ &= (x^2+1)(x^2+4)\left(\frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}\right) \\ &= (x^2+4)(Ax+B) + (x^2+1)(Cx+D) \\ &= (Ax^3+Bx^2+4Ax+4B) + (Cx^3+Dx^2+Cx+D) \\ &= (A+C)x^3 + (B+D)x^2 + (4A+C)x + (4B+D), \end{aligned}$$

from which we can equate the coefficients to obtain the system of differential equations

$$\begin{aligned} A+C &= 0, \\ B+D &= 1, \\ 4A+C &= 0, \\ 4B+D &= 0, \end{aligned}$$

which we can simultaneously solve to obtain the coefficients

$$\begin{aligned} A &= 0, \\ B &= -\frac{1}{3}, \\ C &= 0, \\ D &= \frac{4}{3}. \end{aligned}$$

So we can write our integrand as

$$\begin{aligned} \frac{x^2}{(x^2+1)(x^2+4)} &= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} \\ &= \frac{0x+(-\frac{1}{3})}{x^2+1} + \frac{0x+\frac{4}{3}}{x^2+4} \\ &= -\frac{1}{3}\frac{1}{x^2+1} + \frac{4}{3}\frac{1}{x^2+4}, \end{aligned}$$

which implies

$$\begin{aligned} \int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)} &= \int_0^\infty -\frac{1}{3}\frac{1}{x^2+1} + \frac{4}{3}\frac{1}{x^2+4}\,dx \\ &= -\frac{1}{3}\int_0^\infty \frac{1}{x^2+1}\,dx + \frac{4}{3}\int_0^\infty \frac{1}{x^2+4}\,dx \\ &= -\frac{1}{3}\int_0^\infty \frac{1}{x^2+1}\,dx + \frac{4}{3}\int_0^\infty \frac{1}{4(\frac{x^2}{4}+1)}\,dx \\ &= -\frac{1}{3}\int_0^\infty \frac{1}{x^2+1}\,dx + \frac{1}{3}\int_0^\infty \frac{1}{1+(\frac{x}{2})^2}\,dx. \end{aligned}$$

Furthermore, by substituting $u = \frac{x}{2}$ and $du = \frac{1}{2}\,dx$, the second term of our latest expression becomes

$$\begin{aligned}\frac{1}{3}\int_0^\infty \frac{1}{1+(\frac{x}{2})^2}\,dx &= \frac{1}{3}\int_{\frac{0}{2}}^{\frac{\infty}{2}} \frac{1}{1+u^2}(2\,du)\\ &= \frac{2}{3}\int_0^\infty \frac{1}{u^2+1}\,du.\end{aligned}$$

Finally, by invoking Exercise 86.1, we obtain

$$\begin{aligned}\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)} &= -\frac{1}{3}\int_0^\infty \frac{1}{x^2+1}\,dx + \frac{1}{3}\int_0^\infty \frac{1}{(\frac{x}{2})^2+1}\,dx\\ &= -\frac{1}{3}\int_0^\infty \frac{1}{x^2+1}\,dx + \frac{2}{3}\int_0^\infty \frac{1}{u^2+1}\,du\\ &= -\frac{1}{3}\left(\frac{\pi}{2}\right) + \frac{2}{3}\left(\frac{\pi}{2}\right) \quad \text{by Exercise 86.1}\\ &= -\frac{\pi}{6} + \frac{\pi}{3}\\ &= \boxed{\frac{\pi}{6}}.\end{aligned}$$

□

Use residues to find the Cauchy principal values of the integrals in Exercises 7 and 8.

86.7. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+2x+2}.$

Solution. The isolated singularities of

$$\frac{1}{z^2+2z+2} = \frac{1}{(z-(-1-i))(z-(-1+i))}$$

are $z = -1 \pm i$, each of which are poles of order 1. But only $z = -1+i$ lies within the upper semicircle C_R with a sufficiently large radius $R > 0$. The residue of $\frac{1}{z^2+2z+2}$ at $z = -1+i$ is

$$\begin{aligned}\operatorname*{Res}_{z=-1+i} \frac{1}{z^2+2z+2} &= \left.\frac{1}{z-(-1-i)}\right|_{z=-1+i}\\ &= \frac{1}{(-1+i)-(-1-i)}\\ &= \frac{1}{2i}\\ &= \frac{1}{2i}\frac{i}{i}\\ &= \frac{i}{2i^2}\\ &= \frac{i}{2(-1)}\\ &= -\frac{i}{2}.\end{aligned}$$

By the residue theorem from Section 77 applied to the region bounded by $[-R, R]$ and C_R, we have

$$\begin{aligned}
\int_{-R}^{R} \frac{dz}{z^2+2z+2} &= 2\pi i \operatorname*{Res}_{z=i} \frac{1}{z^2+2z+2} - \int_{C_R} \frac{dz}{z^2+2z+2} \\
&= 2\pi i \left(-\frac{i}{2}\right) - \int_{C_R} \frac{dz}{z^2+2z+2} \\
&= -\pi i^2 - \int_{C_R} \frac{dz}{z^2+2z+2} \\
&= -\pi(-1) - \int_{C_R} \frac{dz}{z^2+2z+2} \\
&= \pi - \int_{C_R} \frac{dz}{z^2+2z+2}.
\end{aligned}$$

Next, if we substitute $z = Re^{i\theta}$ and $dz = iRe^{i\theta}\, d\theta$ for all $0 < \theta < \pi$, we obtain

$$\begin{aligned}
\left|\int_{C_R} \frac{dz}{z^2+2z+2}\right| &= \left|\int_0^{\pi} \frac{iRe^{i\theta}\, d\theta}{(Re^{i\theta})^2 + 2(Re^{i\theta}) + 2}\right| \\
&= \left|\int_0^{\pi} \frac{iRe^{i\theta}\, d\theta}{R^2e^{2i\theta} + 2Re^{i\theta} + 2}\right| \\
&\le \int_0^{\pi} \frac{|iRe^{i\theta}|}{|R^2e^{2i\theta} + 2Re^{i\theta} + 2|}\, d\theta \\
&\le \int_0^{\pi} \frac{|iRe^{i\theta}|}{||R^2e^{2i\theta}| - |2Re^{i\theta}| - |2||}\, d\theta \\
&= \int_0^{\pi} \frac{R}{|R^2 - 2R - 2|}\, d\theta \\
&= \frac{R}{R^2 - 2R - 2} \int_0^{\pi} 1\, d\theta \\
&= \frac{R}{R^2 - 2R - 2} \theta\Big|_0^{\pi} \\
&= \frac{R}{R^2 - 2R - 2}(\pi - 0) \\
&= \frac{\pi R}{R^2 - 2R - 2},
\end{aligned}$$

which implies

$$\begin{aligned}
0 &\le \left|\lim_{R\to\infty}\int_{C_R}\frac{dz}{z^2+2z+2}\right| \\
&= \lim_{R\to\infty}\left|\int_{C_R}\frac{dz}{z^2+2z+2}\right| \\
&\le \lim_{R\to\infty}\frac{\pi R}{R^2-2R-2} \\
&= \lim_{R\to\infty}\frac{\pi R}{R^2-2R-2}\frac{\frac{1}{R}}{\frac{1}{R}} \\
&= \lim_{R\to\infty}\frac{\pi}{R-2-\frac{2}{R}} \\
&= \frac{\pi}{\infty-2-0} \\
&= 0,
\end{aligned}$$

from which we conclude

$$\lim_{R\to\infty}\int_{C_R}\frac{dz}{z^2+2z+2} = 0.$$

Finally, since $\frac{1}{x^2+2x+2}$ is an even function, we conclude

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{dx}{x^2+2x+2} &= \lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{x^2+2x+2} \\
&= \lim_{R\to\infty}\left(\pi-\int_{C_R}\frac{dz}{z^2+2z+2}\right) \\
&= \lim_{R\to\infty}\pi-\lim_{R\to\infty}\int_{C_R}\frac{dz}{z^2+2z+2} \\
&= \pi-0 \\
&= \boxed{\pi}.
\end{aligned}$$

□

86.8. $\displaystyle\int_{-\infty}^{\infty}\frac{x\,dx}{(x^2+1)(x^2+2x+2)}$.

Solution. Although one can compute this exercise by applying the residue theorem directly, as we have done in the previous exercises, it is possible to use only methods from first-year calculus where applicable while directly invoking Exercises 86.1 and 86.7. With that in mind, if we employ the method of decomposition by partial fractions, we can write our integrand as

$$\frac{x}{(x^2+1)(x^2+2x+2)} = \frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2x+2},$$

which implies

$$
\begin{aligned}
x &= (x^2+1)(x^2+2x+2)\frac{x}{(x^2+1)(x^2+2x+2)} \\
&= (x^2+1)(x^2+4)\left(\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2x+2}\right) \\
&= (x^2+2x+2)(Ax+B)+(x^2+1)(Cx+D) \\
&= (Ax^3+2Ax^2+2Ax+Bx^2+2Bx+2B)+(Cx^3+Dx^2+Cx+D) \\
&= (A+C)x^3+(2A+B+D)x^2+(2A+2B+C)x+(2B+D),
\end{aligned}
$$

from which we can equate the coefficients to obtain the system of differential equations

$$
\begin{aligned}
A+C &= 0, \\
2A+B+D &= 1, \\
2A+2B+C &= 0, \\
2B+D &= 0,
\end{aligned}
$$

which we can simultaneously solve to obtain the coefficients

$$
\begin{aligned}
A &= \frac{1}{5}, \\
B &= \frac{2}{5}, \\
C &= -\frac{1}{5}, \\
D &= -\frac{4}{5}.
\end{aligned}
$$

So we can write our integrand as

$$
\begin{aligned}
\frac{x}{(x^2+1)(x^2+2x+2)} &= \frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2x+2} \\
&= \frac{\frac{1}{5}x+\frac{2}{5}}{x^2+1}+\frac{-\frac{1}{5}x+(-\frac{4}{5})}{x^2+2x+2} \\
&= \frac{1}{5}\frac{x}{x^2+1}+\frac{2}{5}\frac{1}{x^2+1}-\frac{1}{5}\frac{x}{x^2+2x+2}-\frac{4}{5}\frac{1}{x^2+2x+2} \\
&= \frac{1}{5}\frac{x}{x^2+1}+\frac{2}{5}\frac{1}{x^2+1}-\frac{1}{10}\frac{2x}{x^2+2x+2}-\frac{4}{5}\frac{1}{x^2+2x+2} \\
&= \frac{1}{5}\frac{x}{x^2+1}+\frac{2}{5}\frac{1}{x^2+1}-\frac{1}{10}\frac{2x+2-2}{x^2+2x+2}-\frac{4}{5}\frac{1}{x^2+2x+2} \\
&= \frac{1}{5}\frac{x}{x^2+1}+\frac{2}{5}\frac{1}{x^2+1}-\frac{1}{10}\left(\frac{2x+2}{x^2+2x+2}-\frac{2}{x^2+2x+2}\right)-\frac{4}{5}\frac{1}{x^2+2x+2} \\
&= \frac{1}{5}\frac{x}{x^2+1}+\frac{2}{5}\frac{1}{x^2+1}-\frac{1}{10}\frac{2x+2}{x^2+2x+2}+\frac{1}{5}\frac{1}{x^2+2x+2}-\frac{4}{5}\frac{1}{x^2+2x+2} \\
&= \frac{1}{5}\frac{x}{x^2+1}+\frac{2}{5}\frac{1}{x^2+1}-\frac{1}{10}\frac{2x+2}{x^2+2x+2}-\frac{3}{5}\frac{1}{x^2+2x+2},
\end{aligned}
$$

which implies

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)} &= \int_{0}^{\infty} \frac{1}{5}\frac{x}{x^2+1} + \frac{2}{5}\frac{1}{x^2+1} - \frac{1}{10}\frac{2x+2}{x^2+2x+2} - \frac{3}{5}\frac{1}{x^2+2x+2}\,dx \\
&= \frac{1}{5}\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx + \frac{2}{5}\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx \\
&\quad - \frac{1}{10}\int_{-\infty}^{\infty} \frac{2x+2}{x^2+2x+2}\,dx - \frac{3}{5}\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}\,dx.
\end{aligned}$$

Furthermore, by substituting $u = x^2$ and $du = 2x\,dx$, the first term of our latest expression becomes

$$\begin{aligned}
\frac{1}{5}\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx &= \frac{1}{5}\lim_{a\to\infty}\int_{-a}^{a} \frac{x}{x^2+1}\,dx \\
&= \frac{1}{5}\lim_{a\to\infty}\int_{(-a)^2}^{a^2} \frac{1}{u}\frac{du}{2} \\
&= \frac{1}{10}\lim_{a\to\infty}\int_{a^2}^{a^2} \frac{1}{u}\,du \\
&= \frac{1}{10}\lim_{a\to\infty} \ln(|u|)\Big|_{a^2}^{a^2} \\
&= \frac{1}{10}\lim_{a\to\infty} (\ln(|a^2|) - \ln(|a^2|)) \\
&= \frac{1}{10}\lim_{a\to\infty} \ln\left(\frac{|a^2|}{|a^2|}\right) \\
&= \frac{1}{10}\lim_{a\to\infty} \ln(1) \\
&= \frac{1}{10}\lim_{a\to\infty} 0 \\
&= \frac{1}{10}0 \\
&= 0.
\end{aligned}$$

Likewise, by substituting $u = x^2 + 2x + 2$ and $du = 2x + 2\,dx$, the third term of our latest expression becomes

$$
\begin{aligned}
-\frac{1}{10}\frac{2x+2}{x^2+2x+2} &= -\frac{1}{10}\lim_{a\to\infty}\int_{-a}^{a}\frac{2x+2}{x^2+2x+2}\,dx \\
&= -\frac{1}{10}\lim_{a\to\infty}\int_{(-a)^2+2(-a)+2}^{a^2+2a+2}\frac{1}{u}\,du \\
&= -\frac{1}{10}\lim_{a\to\infty}\int_{a^2-2a+2}^{a^2+2a+2}\frac{1}{u}\,du \\
&= -\frac{1}{10}\lim_{a\to\infty}\ln(|u|)\Big|_{a^2-2a+2}^{a^2+2a+2} \\
&= -\frac{1}{10}\lim_{a\to\infty}(\ln(|a^2-2a+2|)-\ln(|a^2+2a+2|) \\
&= -\frac{1}{10}\lim_{a\to\infty}\ln\left(\frac{|a^2-2a+2|}{|a^2+2a+2|}\right) \\
&= -\frac{1}{10}\lim_{a\to\infty}\ln\left(\frac{|a^2-2a+2|\frac{1}{|a^2|}}{|a^2+2a+2|\frac{1}{|a^2|}}\right) \\
&= -\frac{1}{10}\lim_{a\to\infty}\ln\left(\frac{|1-\frac{2}{a}+\frac{2}{a^2}|}{|1+\frac{2}{a}+\frac{2}{a^2}|}\right) \\
&= -\frac{1}{10}\lim_{a\to\infty}\ln\left(\frac{|1-0+0|}{|1+0+0|}\right) \\
&= -\frac{1}{10}\lim_{a\to\infty}\ln(1) \\
&= -\frac{1}{10}\lim_{a\to\infty}0 \\
&= -\frac{1}{10}0 \\
&= 0.
\end{aligned}
$$

Finally, since $\frac{x}{x^2+1}$ is an even function, the second term of our latest expression becomes

$$
\begin{aligned}
\frac{2}{5}\int_{-\infty}^{\infty}\frac{1}{x^2+1}\,dx &= \frac{2}{5}\left(2\int_{0}^{\infty}\frac{1}{x^2+1}\,dx\right) \\
&= \frac{4}{5}\int_{0}^{\infty}\frac{1}{x^2+1}\,dx.
\end{aligned}
$$

So our latest expression further reduces to

$$
\begin{aligned}
\int_{-\infty}^{\infty}\frac{x\,dx}{(x^2+1)(x^2+2x+2)} &= \frac{1}{5}\int_{-\infty}^{\infty}\frac{x}{x^2+1}\,dx+\frac{2}{5}\int_{-\infty}^{\infty}\frac{1}{x^2+1}\,dx \\
&\quad -\frac{1}{10}\int_{-\infty}^{\infty}\frac{2x+2}{x^2+2x+2}\,dx-\frac{3}{5}\int_{-\infty}^{\infty}\frac{1}{x^2+2x+2}\,dx \\
&= 0+\frac{4}{5}\int_{0}^{\infty}\frac{1}{x^2+1}\,dx-0-\frac{3}{5}\int_{-\infty}^{\infty}\frac{1}{x^2+2x+2}\,dx \\
&= \frac{4}{5}\int_{0}^{\infty}\frac{1}{x^2+1}\,dx-\frac{3}{5}\int_{-\infty}^{\infty}\frac{1}{x^2+2x+2}\,dx.
\end{aligned}
$$

Finally, by invoking Exercise 86.1 and Exercise 86.7, we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)} &= \frac{4}{5}\int_0^{\infty} \frac{1}{x^2+1}\,dx - \frac{3}{5}\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}\,dx \\
&= \frac{4}{5}\left(\frac{\pi}{2}\right) - \frac{3}{5}(\pi) \quad \text{by Exercises 86.1 and 86.7} \\
&= \frac{2\pi}{5} - \frac{3\pi}{5} \\
&= \boxed{-\frac{\pi}{5}}.
\end{aligned}$$

□