MATH 165B discussion
University of California, Riverside

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Spring 2022

Solutions to suggested homework problems from
Complex Variables and Applications, Ninth Edition by James Brown and Ruel Churchill
Homework 6: Section 92, Exercises 1, 2, 4, 5

Use residues to establish the following integration formulas:

92.1. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}.$

Solution. By applying the parametric representation $z = e^{i\theta}$, we can use the substitutions

$$\sin z = \frac{z - z^{-1}}{2i}, \quad d\theta = \frac{dz}{iz}.$$

So we can write our integral as

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} &= \int_{|z|=1} \frac{1}{5+4\frac{z-z^{-1}}{2i}} \frac{dz}{iz} \\
&= \int_{|z|=1} \frac{1}{5+4\frac{z-z^{-1}}{2i}} \frac{dz}{iz} \\
&= \int_{|z|=1} \frac{1}{5+4\frac{z-z^{-1}}{2i}} \frac{2}{2} \frac{dz}{iz} \\
&= 2\int_{|z|=1} \frac{1}{2iz(5+4\frac{z-z^{-1}}{2i})}\, dz \\
&= 2\int_{|z|=1} \frac{1}{10iz+8z^2-8}\, dz \\
&= 2\int_{|z|=1} \frac{1}{4(\frac{5}{2}iz+z^2-1)}\, dz \\
&= \frac{1}{2}\int_{|z|=1} \frac{1}{z^2+\frac{5}{2}iz-1}\, dz.
\end{aligned}$$

The isolated singularities of

$$z^2 + \frac{5}{2}iz - 1 = (z+2i)\left(z + \frac{i}{2}\right)$$

are $z = 2i$ and $z = -\frac{i}{2}$, both of which are poles of order 1. However, only the singularity

$z = -\frac{i}{2}$ lies inside the circle $|z| = 1$. We have

$$\begin{aligned}
\operatorname*{Res}_{z=-\frac{i}{2}} \frac{1}{z^2 + \frac{5}{2}iz - 1} &= \left.\frac{1}{z+2i}\right|_{z=-\frac{i}{2}} \\
&= \frac{1}{-\frac{i}{2} + 2i} \\
&= \frac{1}{\frac{3i}{2}} \\
&= \frac{1}{\frac{3i}{2}} \frac{2i}{2i} \\
&= \frac{2i}{3i^2} \\
&= \frac{2i}{3(-1)} \\
&= -\frac{2i}{3}.
\end{aligned}$$

By the residue theorem from Section 76, we conclude

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} &= \frac{1}{2} \int_{|z|=1} \frac{z}{z^2 + \frac{5}{2}iz - 1}\, dz \\
&= \frac{1}{2}\left(2\pi i \operatorname*{Res}_{z=-\frac{i}{2}} \frac{z}{z^2 + \frac{5}{2}iz - 1}\right) \\
&= \frac{1}{2}\left(2\pi i\left(-\frac{2i}{3}\right)\right) \\
&= -\frac{2\pi i^2}{3} \\
&= -\frac{2\pi(-1)}{3} \\
&= \boxed{\frac{2\pi}{3}}.
\end{aligned}$$

□

92.2. $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + \sin^2\theta} = \sqrt{2}\pi.$

Solution. By applying the parametric representation $z = e^{i\theta}$, we can use the substitutions

$$\sin z = \frac{z - z^{-1}}{2i}, \quad d\theta = \frac{dz}{iz}.$$

So we can write our integral as

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta} &= \int_{|z|=1} \frac{1}{1+(\frac{z-z^{-1}}{2i})^2} \frac{dz}{iz} \\
&= \int_{|z|=1} \frac{1}{1+\frac{(z-z^{-1})^2}{4i^2}} \frac{dz}{iz} \\
&= \int_{|z|=1} \frac{1}{1+\frac{z^2-2+z^{-2}}{4(-1)}} \frac{4iz^2}{4iz^2} \frac{dz}{iz} \\
&= 4i \int_{|z|=1} \frac{z}{1-\frac{z^2-2+z^{-2}}{4}} \frac{dz}{4i^2 z^2} \\
&= 4i \int_{|z|=1} \frac{z}{4z^2-4z^2(\frac{z^2-2+z^{-2}}{4})} \frac{dz}{-1} \\
&= 4i \int_{|z|=1} \frac{z}{-4z^2+(z^4-2z^2+1)}\, dz \\
&= 4i \int_{|z|=1} \frac{z}{z^4-6z^2+1}\, dz.
\end{aligned}$$

The isolated singularities of

$$\begin{aligned}
\frac{z}{z^4-6z^2+1} &= \frac{z}{(z^2-(3-2\sqrt{2}))(z^2-(3+2\sqrt{2}))} \\
&= \frac{z}{(z+\sqrt{3-2\sqrt{2}})(z-\sqrt{3-2\sqrt{2}})(z+\sqrt{3+2\sqrt{2}})(z-\sqrt{3+2\sqrt{2}})}
\end{aligned}$$

are $z = \pm\sqrt{3 \pm 2\sqrt{2}}$, each of which are poles of order 1. However, only the singularities $z = \pm\sqrt{3-2\sqrt{2}}$ lie inside the circle $|z| = 1$. We have

$$\begin{aligned}
\operatorname*{Res}_{z=-\sqrt{3-2\sqrt{2}}} \frac{z}{z^4-6z^2+1} &= \left.\frac{z}{(z-\sqrt{3-2\sqrt{2}})(z+\sqrt{3+2\sqrt{2}})(z-\sqrt{3+2\sqrt{2}})}\right|_{z=-\sqrt{3-2\sqrt{2}}} \\
&= \left.\frac{z}{(z-\sqrt{3-2\sqrt{2}})(z^2-(3+2\sqrt{2}))}\right|_{z=-\sqrt{3-2\sqrt{2}}} \\
&= \frac{-\sqrt{3-2\sqrt{2}}}{((-\sqrt{3-2\sqrt{2}})-\sqrt{3-2\sqrt{2}})((-\sqrt{3-2\sqrt{2}})^2-(3+2\sqrt{2}))} \\
&= \frac{-\sqrt{3-2\sqrt{2}}}{(-2\sqrt{3-2\sqrt{2}})(3-2\sqrt{2}-3-2\sqrt{2})} \\
&= \frac{1}{2(-4\sqrt{2})} \\
&= -\frac{1}{8\sqrt{2}}
\end{aligned}$$

and

$$
\begin{aligned}
\operatorname*{Res}_{z=\sqrt{3-2\sqrt{2}}} \frac{z}{z^4-6z^2+1} &= \left.\frac{z}{(z+\sqrt{3-2\sqrt{2}})(z+\sqrt{3+2\sqrt{2}})(z-\sqrt{3+2\sqrt{2}})}\right|_{z=\sqrt{3-2\sqrt{2}}} \\
&= \left.\frac{z}{(z+\sqrt{3-2\sqrt{2}})(z^2-(3+2\sqrt{2}))}\right|_{z=\sqrt{3-2\sqrt{2}}} \\
&= \frac{\sqrt{3-2\sqrt{2}}}{(2\sqrt{3-2\sqrt{2}})(3-2\sqrt{2}-3-2\sqrt{2}))} \\
&= \frac{\sqrt{3-2\sqrt{2}}}{(2\sqrt{3-2\sqrt{2}})(3-2\sqrt{2}-3-2\sqrt{2})} \\
&= \frac{1}{2(-4\sqrt{2})} \\
&= -\frac{1}{8\sqrt{2}}.
\end{aligned}
$$

By the residue theorem from Section 76, we conclude

$$
\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} &= 4i \int_{|z|=1} \frac{z}{z^4-6z^2+1}\,dz \\
&= 4i\left(2\pi i\left(\operatorname*{Res}_{z=-\sqrt{3-2\sqrt{2}}} \frac{z}{z^4-6z^2+1} + \operatorname*{Res}_{z=\sqrt{3-2\sqrt{2}}} \frac{z}{z^4-6z^2+1}\right)\right) \\
&= 8\pi i^2\left(\left(-\frac{1}{8\sqrt{2}}\right)+\left(-\frac{1}{8\sqrt{2}}\right)\right) \\
&= 8\pi(-1)\left(-\frac{1}{4\sqrt{2}}\right) \\
&= \frac{2\pi}{\sqrt{2}} \\
&= \boxed{\sqrt{2}\pi}.
\end{aligned}
$$

□

92.4. $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}} \quad (-1 < a < 1).$

Solution. By applying the parametric representation $z = e^{i\theta}$, we can use the substitutions

$$
\cos z = \frac{z+z^{-1}}{2}, \quad d\theta = \frac{dz}{iz}.
$$

So we can write our integral as

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} &= \int_{|z|=1} \frac{1}{1 + a\frac{z+z^{-1}}{2}} \frac{dz}{iz} \\
&= \int_{|z|=1} \frac{1}{1 + a\frac{z+z^{-1}}{2}} \frac{2i}{2i} \frac{dz}{iz} \\
&= \int_{|z|=1} \frac{2i}{2 + a(z + z^{-1})} \frac{dz}{i^2 z} \\
&= \int_{|z|=1} \frac{2i}{2z + a(z^2 + 1)} \frac{dz}{-1} \\
&= -2i \int_{|z|=1} \frac{1}{a(\frac{2}{a}z + z^2 + 1)}\, dz \\
&= -\frac{2i}{a} \int_{|z|=1} \frac{1}{z^2 + \frac{2}{a}z + 1}\, dz.
\end{aligned}$$

The isolated singularities of

$$\frac{1}{z^2 + \frac{2}{a}z + 1} = \frac{1}{(z - \frac{-1-\sqrt{1-a^2}}{a})(z - \frac{-1+\sqrt{1-a^2}}{a})}$$

are $z = \frac{-1\pm\sqrt{1-a^2}}{a}$, both of which are poles of order 1. However, only the singularity $z = \frac{-1+\sqrt{1-a^2}}{a}$ lies inside the circle $|z| = 1$ for all $-1 < a < 1$. We have

$$\begin{aligned}
\operatorname*{Res}_{z=\frac{-1+\sqrt{1-a^2}}{a}} \frac{1}{z^2 + \frac{2}{a}z + 1} &= \left. \frac{1}{z - \frac{-1-\sqrt{1-a^2}}{a}} \right|_{z=\frac{-1+\sqrt{1-a^2}}{a}} \\
&= \frac{1}{\frac{-1+\sqrt{1-a^2}}{a} - \frac{-1-\sqrt{1-a^2}}{a}} \\
&= \frac{1}{\frac{-1+\sqrt{1-a^2}+1+\sqrt{1-a^2}}{a}} \\
&= \frac{1}{\frac{2\sqrt{1-a^2}}{a}} \\
&= \frac{a}{2\sqrt{1-a^2}}.
\end{aligned}$$

By the residue theorem from Section 76, we conclude

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} &= -\frac{2i}{a} \int_{|z|=1} \frac{1}{z^2 + \frac{2}{a}z + 1}\, dz \\
&= -\frac{2i}{a}\left(2\pi i \operatorname*{Res}_{z=-a+\sqrt{a^2-1}} \frac{1}{z^2 + \frac{2}{a}z + 1}\right) \\
&= -\frac{4\pi i^2}{a} \operatorname*{Res}_{z=-a+\sqrt{a^2-1}} \frac{1}{z^2 + \frac{2}{a}z + 1} \\
&= -\frac{4\pi(-1)}{a} \frac{a}{2\sqrt{1-a^2}} \\
&= \boxed{\frac{2\pi}{\sqrt{1-a^2}}}
\end{aligned}$$

for all $-1 < a < 1$. □

92.5. $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a + \cos\theta)^2} = \frac{2a\pi}{(\sqrt{a^2-1})^3} \quad (a > 1).$

The textbook made a typo of their final answer, as it forgot to include the factor of 2 in the numerator. I entered this integral for $a = 6$ in WolframAlpha (https://www.wolframalpha.com/input?i=%5Cint_0%5E%7B2%5Cpi%7D+%5Cfrac%7Bdx%7D%7B%286%2Bcos+x%29%5E2%7D); this input matches my boxed answer in my solution for $a = 6$ instead of the textbook's printed answer for $a = 6$.

Solution. By applying the parametric representation $z = e^{i\theta}$, we can use the substitutions

$$\cos z = \frac{z + z^{-1}}{2}, \qquad d\theta = \frac{dz}{iz}.$$

So we can write our integral as

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{(a + \cos\theta)^2} &= \int_{|z|=1} \frac{1}{(a + \frac{z+z^{-1}}{2})^2} \frac{dz}{iz} \\
&= \int_{|z|=1} \frac{1}{(a + \frac{z+z^{-1}}{2})^2} \frac{4iz}{4iz} \frac{dz}{iz} \\
&= \int_{|z|=1} \frac{4iz}{(a + \frac{z+z^{-1}}{2})^2} \frac{dz}{i^2(2z)^2} \\
&= \int_{|z|=1} \frac{4iz}{(2z(a + \frac{z+z^{-1}}{2}))^2} \frac{dz}{-1} \\
&= -4i \int_{|z|=1} \frac{z}{(2az + z^2 + 1)^2}\, dz \\
&= -4i \int_{|z|=1} \frac{z}{(z^2 + 2az + 1)^2}\, dz.
\end{aligned}$$

The isolated singularities of

$$\frac{z}{(z^2 + 2az + 1)^2} = \frac{z}{(z - (-a - \sqrt{a^2-1}))^2 (z - (-a + \sqrt{a^2-1}))^2}$$

are $z = -a \pm \sqrt{a^2 - 1}$, both of which are poles of order 2. However, only the singularity $z = -a + \sqrt{a^2 - 1}$ lies inside the circle $|z| = 1$ for all $a > 1$. We have

$$
\begin{aligned}
&\operatorname*{Res}_{z=-a+\sqrt{a^2-1}} \frac{z}{(z^2 + 2az + 1)^2} \\
&= \frac{1}{(2-1)!} \frac{d^{(2-1)}}{dz^{(2-1)}} \frac{z}{(z - (-a - \sqrt{a^2 - 1}))^2}\bigg|_{z=-a+\sqrt{a^2-1}} \\
&= \frac{d}{dz} \frac{z}{(z - (-a - \sqrt{a^2 - 1}))^2}\bigg|_{z=-a+\sqrt{a^2-1}} \\
&= \frac{\frac{d}{dz}(z)(z - (-a - \sqrt{a^2 - 1}))^2 - (z)\frac{d}{dz}(z - (-a - \sqrt{a^2 - 1}))^2}{(z - (-a - \sqrt{a^2 - 1}))^4}\bigg|_{z=-a+\sqrt{a^2-1}} \\
&= \frac{1(z - (-a - \sqrt{a^2 - 1}))^2 - z(2(z - (-a - \sqrt{a^2 - 1}))\frac{d}{dz}(z - (-a - \sqrt{a^2 - 1})))}{(z - (-a - \sqrt{a^2 - 1}))^4}\bigg|_{z=-a+\sqrt{a^2-1}} \\
&= \frac{(z - (-a - \sqrt{a^2 - 1}))^2 - z(2(z - (-a - \sqrt{a^2 - 1}))1))}{(z - (-a - \sqrt{a^2 - 1}))^4}\bigg|_{z=-a+\sqrt{a^2-1}} \\
&= \frac{(z - (-a - \sqrt{a^2 - 1}))^2 - 2z(z - (-a - \sqrt{a^2 - 1}))}{(z - (-a - \sqrt{a^2 - 1}))^4}\bigg|_{z=-a+\sqrt{a^2-1}} \\
&= \frac{(z - (-a - \sqrt{a^2 - 1}))((z - (-a - \sqrt{a^2 - 1})) - 2z)}{(z - (-a - \sqrt{a^2 - 1}))^4}\bigg|_{z=-a+\sqrt{a^2-1}} \\
&= \frac{(z - (-a - \sqrt{a^2 - 1})) - 2z}{(z - (-a - \sqrt{a^2 - 1}))^3}\bigg|_{z=-a+\sqrt{a^2-1}} \\
&= \frac{-z + a + \sqrt{a^2 - 1}}{(z + a + \sqrt{a^2 - 1}))^3}\bigg|_{z=-a+\sqrt{a^2-1}} \\
&= \frac{-(-a + \sqrt{a^2 - 1}) + a + \sqrt{a^2 - 1}}{((-a + \sqrt{a^2 - 1}) + a + \sqrt{a^2 - 1}))^3} \\
&= \frac{a - \sqrt{a^2 - 1} + a + \sqrt{a^2 - 1}}{(-a + \sqrt{a^2 - 1} + a + \sqrt{a^2 - 1}))^3} \\
&= \frac{2a}{(2\sqrt{a^2 - 1}))^3} \\
&= \frac{2a}{8(\sqrt{a^2 - 1})^3} \\
&= \frac{a}{4(\sqrt{a^2 - 1})^3}.
\end{aligned}
$$

By the residue theorem from Section 76, we conclude

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} &= -4i \int_{|z|=1} \frac{z}{(z^2+2az+1)^2}\, dz \\
&= -4i \left(2\pi i \operatorname*{Res}_{z=-a+\sqrt{a^2-1}} \frac{z}{(z^2+2az+1)^2} \right) \\
&= -8\pi i^2 \operatorname*{Res}_{z=-a+\sqrt{a^2-1}} \frac{z}{(z^2+2az+1)^2} \\
&= -8\pi(-1) \frac{a}{4(\sqrt{a^2-1})^3} \\
&= \boxed{\frac{2a\pi}{(\sqrt{a^2-1})^3}}
\end{aligned}$$

for all $a > 1$. □