MATH 165B discussion
University of California, Riverside

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Spring 2022

Solutions to suggested homework problems from
Complex Variables and Applications, Ninth Edition by James Brown and Ruel Churchill
Homework 7: Section 88, Exercises 2, 3, 4, 5, 8, 9

Use residues to derive the integration formulas in Exercises 1 through 5.

88.2. $\displaystyle\int_0^\infty \frac{\cos(ax)}{x^2+1}\,dx = \frac{\pi}{2}e^{-a}$ for all $a \in \mathbb{R}$

Solution. The isolated singularities of

$$\frac{e^{i(az)}}{z^2+1} = \frac{e^{i(az)}}{(z+i)(z-i)}$$

are $z = \pm i$, each of which are poles of order 1. But only $z = i$ lies within the upper semicircle C_R with a sufficiently large radius $R > 0$. The residue of $\dfrac{1}{z^2+1}$ at $z = i$ is

$$\begin{aligned}
\operatorname*{Res}_{z=i} \frac{e^{i(az)}}{z^2+1} &= \left.\frac{e^{i(az)}}{z+i}\right|_{z=i} \\
&= \frac{e^{i(ai)}}{i+1} \\
&= \frac{e^{ai^2}}{2i} \\
&= \frac{e^{a(-1)}}{2i} \\
&= \frac{e^{-a}}{2i} \\
&= \frac{e^{-a}}{2i}\frac{i}{i} \\
&= \frac{e^{-a}i}{2i^2} \\
&= \frac{e^{-a}i}{2(-1)} \\
&= -\frac{e^{-a}i}{2}.
\end{aligned}$$

By the residue theorem from Section 77 applied to the closed region bounded by $[-R, R]$

and C_R in the counterclockwise sense, we have

$$\begin{aligned}
\int_{-R}^{R} \frac{e^{i(ax)}}{x^2+1}\,dx &= 2\pi i \operatorname*{Res}_{z=bi} \frac{e^{i(az)}}{z^2+1} - \int_{C_R} \frac{e^{i(az)}}{z^2+1}\,dz \\
&= 2\pi i \left(-\frac{e^{-a} i}{2}\right) - \int_{C_R} \frac{e^{i(az)}}{z^2+1}\,dz \\
&= -\pi e^{-a} i^2 - \int_{C_R} \frac{e^{i(az)}}{z^2+1}\,dz \\
&= -\pi e^{-a}(-1) - \int_{C_R} \frac{e^{i(az)}}{z^2+1}\,dz \\
&= \pi e^{-a} - \int_{C_R} \frac{e^{i(az)}}{z^2+1}\,dz.
\end{aligned}$$

Next, if we substitute $z = Re^{i\theta}$ and $dz = iRe^{i\theta}\,d\theta$ for all $0 < \theta < \pi$, we obtain

$$\begin{aligned}
\left|\int_{C_R} \frac{e^{i(az)}}{z^2+1}\,dz\right| &= \left|\int_0^{\pi} \frac{e^{i(aRe^{i\theta})}}{(Re^{i\theta})^2+1}\,d\theta\right| \\
&= \left|\int_0^{\pi} \frac{e^{i(aR(\cos\theta + i\sin\theta))}}{R^2e^{2i\theta}+1} iRe^{i\theta}\,d\theta\right| \\
&= \left|\int_0^{\pi} \frac{e^{iaR\cos\theta + i^2 aR\sin\theta}}{R^2e^{2i\theta}+1} iRe^{i\theta}\,d\theta\right| \\
&= \left|\int_0^{\pi} \frac{e^{iaR\cos\theta - aR\sin\theta}}{R^2e^{2i\theta}+1} iRe^{i\theta}\,d\theta\right| \\
&= \left|\int_0^{\pi} \frac{e^{iaR\cos\theta}e^{-aR\sin\theta}}{R^2e^{2i\theta}+1} iRe^{i\theta}\,d\theta\right| \\
&\leq \int_0^{\pi} \frac{|e^{iaR\cos\theta}e^{-aR\sin\theta}|}{||R^2e^{2i\theta}| - |1||}\,|iRe^{i\theta}\,d\theta| \\
&= \int_0^{\pi} \frac{e^{-aR\sin\theta}}{|R^2-1|} R\,d\theta \\
&= \frac{Re^{-aR\sin\theta}}{R^2-1} \int_0^{\pi} 1\,d\theta \\
&= \frac{Re^{-aR\sin\theta}}{R^2-1} \theta\Big|_0^{\pi} \\
&= \frac{Re^{-aR\sin\theta}}{R^2-1}(\pi - 0) \\
&= \frac{\pi Re^{-aR\sin\theta}}{R^2-1},
\end{aligned}$$

which implies

$$\begin{aligned}
0 &\leq \left| \lim_{R\to\infty} \int_{C_R} \frac{e^{i(az)}}{z^2+1}\,dz \right| \\
&= \lim_{R\to\infty} \left| \int_{C_R} \frac{e^{i(az)}}{z^2+1}\,dz \right| \\
&\leq \lim_{R\to\infty} \frac{\pi R e^{-aR\sin\theta}}{R^2-1} \\
&= \lim_{R\to\infty} \frac{\pi R e^{-aR\sin\theta}}{R^2-1} \frac{\frac{1}{R}}{\frac{1}{R}} \\
&= \lim_{R\to\infty} \frac{\pi e^{-aR\sin\theta}}{R-\frac{1}{R}} \\
&= \frac{\pi \cdot 0}{\infty - 0} \\
&= 0,
\end{aligned}$$

from which we conclude

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{i(az)}}{z^2+1}\,dz = 0.$$

So we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{i(ax)}}{x^2+1}\,dx &= \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i(ax)}}{x^2+1}\,dx \\
&= \lim_{R\to\infty} \left(\pi e^{-a} - \int_{C_R} \frac{e^{i(az)}}{z^2+1}\,dz \right) \\
&= \lim_{R\to\infty} \pi e^{-a} - \lim_{R\to\infty} \int_{C_R} \frac{e^{i(az)}}{z^2+1}\,dz \\
&= \pi e^{-a} - 0 \\
&= \pi e^{-a} + 0i.
\end{aligned}$$

At the same time, Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{i(ax)}}{x^2+1}\,dx &= \int_{-\infty}^{\infty} \frac{\cos(ax) + i\sin(ax)}{x^2+1}\,dx \\
&= \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1} + \frac{i\sin(ax)}{x^2+1}\,dx \\
&= \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1}\,dx + i \int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2+1}\,dx.
\end{aligned}$$

Our two expressions of $\displaystyle\int_{-\infty}^{\infty} \frac{e^{i(ax)}}{x^2+1}\,dx$ allow us to conclude

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1}\,dx + i \int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2+1}\,dx = \pi e^{-a} + 0i,$$

from which we can equate the real components to conclude

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1}\,dx = \pi e^{-a}.$$

Finally, since $\dfrac{\cos(ax)}{x^2+1}$ is an even function, we conclude

$$\begin{aligned}\int_0^\infty \frac{\cos(ax)}{x^2+1}\,dx &= \frac{1}{2}\int_{-\infty}^\infty \frac{\cos(ax)}{x^2+1}\,dx\\ &= \frac{1}{2}(\pi e^{-a})\\ &= \boxed{\frac{\pi}{2}e^{-a}}\end{aligned}$$

for all $a \in \mathbb{R}$. □

88.3. $\displaystyle\int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}\,dx = \frac{\pi}{4b^3}(1+ab)e^{-ab}$ for all $a \in \mathbb{R}$ and $b \in \mathbb{R}\setminus\{0\}$

Solution. The isolated singularities of

$$\begin{aligned}\frac{e^{i(az)}}{(z^2+b^2)^2} &= \frac{e^{i(az)}}{((z+bi)(z-bi))^2}\\ &= \frac{e^{i(az)}}{(z+bi)^2(z-bi)^2}\end{aligned}$$

are $z = \pm bi$, each of which are poles of order 2. But only $z = bi$ lies within the upper semicircle C_R with a sufficiently large radius $R > 0$. The residue of $\dfrac{e^{i(az)}}{(z^2+b^2)^2}$ at $z = bi$

is

$$
\begin{aligned}
\operatorname*{Res}_{z=bi} \frac{e^{i(az)}}{(z^2+b^2)^2} &= \frac{1}{(2-1)!}\left(\frac{e^{i(az)}}{(z+bi)^2}\right)^{(2-1)}\Bigg|_{z=bi} \\
&= \left(\frac{e^{i(az)}}{(z+bi)^2}\right)'\Bigg|_{z=bi} \\
&= \frac{(e^{i(az)})'((z+bi)^2) - (e^{i(az)})((z+bi)^2)'}{(z+bi)^4}\Bigg|_{z=bi} \\
&= \frac{(iae^{i(az)})((z+bi)^2) - (e^{i(az)})(2(z+bi))}{(z+bi)^4}\Bigg|_{z=bi} \\
&= \frac{e^{i(az)}(z+bi)((ia)(z+bi)-2))}{(z+bi)^4}\Bigg|_{z=bi} \\
&= \frac{e^{i(az)}(ia(z+bi)-2)}{(z+bi)^3}\Bigg|_{z=bi} \\
&= \frac{e^{i(a(bi))}(ia(bi+bi)-2)}{(bi+bi)^3} \\
&= \frac{e^{i(a(bi))}(ia(2bi)-2)}{(2bi)^3} \\
&= \frac{e^{abi^2}(2abi^2-2)}{8b^3i^3} \\
&= \frac{e^{ab(-1)}(2ab(-1)-2)}{8b^3(-i)} \\
&= \frac{e^{-ab}(-2ab-2)}{-8b^3i} \\
&= \frac{-2e^{-ab}(ab+1)}{-8b^3i} \\
&= \frac{e^{-ab}(ab+1)}{4b^3i} \\
&= \frac{(1+ab)e^{-ab}}{4b^3i}\,\frac{i}{i} \\
&= \frac{(1+ab)e^{-ab}}{4b^3i^2} \\
&= \frac{(1+ab)e^{-ab}}{4b^3(-1)} \\
&= -\frac{1}{4b^3}(1+ab)e^{-ab}i.
\end{aligned}
$$

By the residue theorem from Section 77 applied to the closed region bounded by $[-R, R]$

and C_R in the counterclockwise sense, we have

$$\begin{aligned}
\int_{-R}^{R} \frac{e^{i(ax)}}{(x^2+b^2)^2}\,dx &= 2\pi i \operatorname*{Res}_{z=bi} \frac{e^{i(az)}}{(z^2+b^2)^2} - \int_{C_R} \frac{e^{i(az)}}{(z^2+b^2)^2}\,dz \\
&= 2\pi i\left(-\frac{1}{4b^3}(1+ab)e^{-ab}i\right) - \int_{C_R} \frac{e^{i(az)}}{(z^2+b^2)^2}\,dz \\
&= -\frac{\pi i^2}{2b^3}(1+ab)e^{-ab} - \int_{C_R} \frac{e^{i(az)}}{(z^2+b^2)^2}\,dz \\
&= -\frac{\pi(-1)}{2b^3}(1+ab)e^{-ab} - \int_{C_R} \frac{e^{i(az)}}{(z^2+b^2)^2}\,dz \\
&= \frac{\pi}{2b^3}(1+ab)e^{-ab} - \int_{C_R} \frac{e^{i(az)}}{(z^2+b^2)^2}\,dz.
\end{aligned}$$

Next, if we substitute $z = Re^{i\theta}$ and $dz = iRe^{i\theta}\,d\theta$ for all $0 < \theta < \pi$, we obtain

$$\begin{aligned}
\left|\int_{C_R} \frac{e^{i(az)}}{(z^2+b^2)^2}\,dz\right| &= \left|\int_0^{\pi} \frac{e^{i(a(Re^{i\theta}))}}{((Re^{i\theta})^2+b^2)^2} iRe^{i\theta}\,d\theta\right| \\
&= \left|\int_0^{\pi} \frac{e^{i(aR(\cos\theta+i\sin\theta))}}{(Re^{i\theta})^4+2(Re^{i\theta})^2b^2+b^4} iRe^{i\theta}\,d\theta\right| \\
&= \left|\int_0^{\pi} \frac{e^{iaR\cos\theta+i^2aR\sin\theta}}{R^4e^{4i\theta}+2R^2e^{2i\theta}b^2+b^4} iRe^{i\theta}\,d\theta\right| \\
&\le \int_0^{\pi} \frac{|e^{iaR\cos\theta-aR\sin\theta}|}{|R^4e^{4i\theta}+2R^2e^{2i\theta}b^2+b^4|}|iRe^{i\theta}|\,d\theta \\
&\le \int_0^{\pi} \frac{|e^{-aR\sin\theta}||e^{iaR\cos\theta}|}{||R^4e^{4i\theta}|-|2R^2e^{2i\theta}b^2|-|b^4||}\,d\theta \\
&= \int_0^{\pi} \frac{e^{-aR\sin\theta}}{|R^4-2R^2b^2-b^4|} R\,d\theta \\
&= \frac{Re^{-aR\sin\theta}}{|R^4-2R^2b^2-b^4|}\int_0^{\pi} 1\,d\theta \\
&= \frac{Re^{-aR\sin\theta}}{|R^4-2R^2b^2-b^4|}\theta\Big|_0^{\pi} \\
&= \frac{Re^{-aR\sin\theta}}{|R^4-2R^2b^2-b^4|}(\pi-0) \\
&= \frac{\pi Re^{-aR\sin\theta}}{|R^4-2R^2b^2-b^4|},
\end{aligned}$$

which implies

$$\begin{aligned}
0 &\le \left|\lim_{R\to\infty}\int_{C_R} \frac{e^{i(az)}}{(z^2+b^2)^2}\,dz\right| \\
&= \lim_{R\to\infty}\left|\int_{C_R} \frac{e^{i(az)}}{(z^2+b^2)^2}\,dz\right| \\
&\le \lim_{R\to\infty} \frac{\pi R e^{-aR\sin\theta}}{|R^4-2R^2b^2-b^4|} \\
&= \lim_{R\to\infty} \frac{\pi R e^{-aR\sin\theta}}{|R^4-2R^2b^2-b^4|}\frac{\frac{1}{R}}{\frac{1}{R}} \\
&= \lim_{R\to\infty} \frac{\pi e^{-aR\sin\theta}}{|R^3-2Rb^2-\frac{b^4}{R}|} \\
&= \lim_{R\to\infty} \frac{\pi e^{-aR\sin\theta}}{R|R^2-2b^2-\frac{b^4}{R^2}|} \\
&= \frac{\pi\cdot 0}{\infty|\infty-2b^2-0|} \\
&= 0,
\end{aligned}$$

from which we conclude

$$\lim_{R\to\infty}\int_{C_R} \frac{e^{i(az)}}{(z^2+b^2)^2}\,dz = 0.$$

So we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{i(ax)}}{(x^2+b^2)^2}\,dx &= \lim_{R\to\infty}\int_{-R}^{R} \frac{e^{i(ax)}}{(x^2+b^2)^2}\,dx \\
&= \lim_{R\to\infty}\left(\frac{\pi}{2b^3}(1+ab)e^{-ab} - \int_{C_R}\frac{e^{i(az)}}{(z^2+b^2)^2}\,dz\right) \\
&= \lim_{R\to\infty}\frac{\pi}{2b^3}(1+ab)e^{-ab} - \lim_{R\to\infty}\int_{C_R}\frac{e^{i(az)}}{(z^2+b^2)^2}\,dz \\
&= \frac{\pi}{2b^3}(1+ab)e^{-ab} - 0 \\
&= \frac{\pi}{2b^3}(1+ab)e^{-ab} + 0i.
\end{aligned}$$

At the same time, Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{i(ax)}}{(x^2+b^2)^2}\,dx &= \int_{-\infty}^{\infty} \frac{\cos(ax)+i\sin(ax)}{(x^2+b^2)^2}\,dx \\
&= \int_{-\infty}^{\infty} \frac{\cos(ax)}{(x^2+b^2)^2} + \frac{i\sin(ax)}{(x^2+b^2)^2}\,dx \\
&= \int_{-\infty}^{\infty} \frac{\cos(ax)}{(x^2+b^2)^2}\,dx + i\int_{-\infty}^{\infty} \frac{\sin(ax)}{(x^2+b^2)^2}\,dx.
\end{aligned}$$

Our two expressions of $\displaystyle\int_{-\infty}^{\infty} \frac{e^{i(ax)}}{(x^2+b^2)^2}\,dx$ allow us to conclude

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{(x^2+b^2)^2}\,dx + i\int_{-\infty}^{\infty} \frac{\sin(ax)}{(x^2+b^2)^2}\,dx = \frac{\pi}{2b^3}(1+ab)e^{-ab} + 0i,$$

from which we can equate the real components to conclude

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{(x^2+b^2)^2}\,dx = \frac{\pi}{2b^3}(1+ab)e^{-ab}.$$

Finally, since $\dfrac{\cos(ax)}{(x^2+b^2)^2}$ is an even function, we conclude

$$\begin{aligned}\int_{0}^{\infty} \frac{\cos(ax)}{(x^2+b^2)^2}\,dx &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(ax)}{(x^2+b^2)^2}\,dx \\ &= \frac{1}{2}\left(\frac{\pi}{2b^3}(1+ab)e^{-ab}\right) \\ &= \boxed{\frac{\pi}{4b^3}(1+ab)e^{-ab}}\end{aligned}$$

for all $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$. □

88.4. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^4+4}\,dx = \frac{\pi}{2}e^{-a}\sin a$ for all $a > 0$

Solution. The isolated singularities of

$$\begin{aligned}\frac{ze^{i(az)}}{z^4+4} &= \frac{ze^{i(az)}}{(z^2+2i)(z^2-2i)} \\ &= \frac{ze^{i(az)}}{(z-(-1+i))(z-(-1-i))(z-(1+i))(z-(1-i))}\end{aligned}$$

are $z = -1 \pm i$ and $z = 1 \pm i$, each of which are poles of order 1. But only $z = -1+i$ and $z = 1+i$ lie within the upper semicircle C_R with a sufficiently large radius $R > 0$. The residue of $\frac{ze^{i(az)}}{z^4+4}$ at $z=-1+i$ is

$$
\begin{aligned}
\operatorname*{Res}_{z=-1+i} \frac{ze^{i(az)}}{z^4+1} &= \left.\frac{ze^{i(az)}}{(z-(-1-i))(z-(1+i))(z-(1-i))}\right|_{z=-1+i} \\
&= \frac{(-1+i)e^{i(a(-1+i))}}{((-1+i)-(-1-i))((-1+i)-(1+i))((-1+i)-(1-i))} \\
&= \frac{(-1+i)e^{-ai+ai^2}}{(2i)(-2)(-2+2i)} \\
&= -\frac{(-1+i)e^{-ai+a(-1)}}{(2i)(-2)(2(-1+i))} \\
&= -\frac{(-1+i)e^{-ai-a}}{-8(-i+i^2)} \\
&= -\frac{(-1+i)e^{-ai}e^{-a}}{8(-i+(-1))} \\
&= -\frac{e^{-a}e^{-ai}(-1+i)}{-8(1+i)} \\
&= \frac{e^{-a}e^{-ai}(-1+i)}{8(1+i)} \\
&= \frac{e^{-a}e^{-ai}(-1+i)}{8(1+i)}\frac{1-i}{1-i} \\
&= \frac{e^{-a}e^{-ai}(-1+2i-i^2)}{8(1-i^2)} \\
&= \frac{e^{-a}e^{-ai}(-1+2i-(-1))}{8(1-(-1))} \\
&= \frac{e^{-a}e^{-ai}(2i)}{8(2)} \\
&= \frac{e^{-a}e^{-ai}i}{8} \\
&= \frac{e^{-a}(\cos(-a)+i\sin(-a))i}{8} \\
&= \frac{e^{-a}(i\cos a - i^2\sin(-a))}{8} \\
&= \frac{e^{-a}(i\cos a-(-1)\sin a)}{8} \\
&= \frac{(e^{-a}\cos a)i + e^{-a}\sin a}{8} \\
&= \frac{e^{-a}\sin a}{8} + \frac{e^{-a}\cos a}{8}i
\end{aligned}
$$

and the residue of $\dfrac{ze^{i(az)}}{z^4+4}$ at $z = 1+i$ is

$$
\begin{aligned}
\operatorname*{Res}_{z=1+i} \frac{ze^{i(az)}}{z^4+4} &= \left.\frac{ze^{i(az)}}{(z-(-1+i))(z-(-1-i))(z-(1-i))}\right|_{z=1+i} \\
&= \frac{(1+i)e^{i(a(1+i))}}{((1+i)-(-1+i))((1+i)-(-1-i))((1+i)-(1-i))} \\
&= \frac{(1+i)e^{ai+ai^2}}{(2)(2+2i)(2i)} \\
&= \frac{(1+i)e^{ai+a(-1)}}{(2)(2(1+i))(2i)} \\
&= \frac{(1+i)e^{ai-a}}{8(1+i)} \\
&= \frac{(1+i)e^{ai}e^{-a}}{8(i+i^2)} \\
&= \frac{e^{-a}e^{ai}(1+i)}{8(i+(-1))} \\
&= \frac{e^{-a}e^{ai}(1+i)}{8(-1+i)}\,\frac{-1-i}{-1-i} \\
&= \frac{e^{-a}e^{ai}(-1-2i-i^2)}{8(1-i^2)} \\
&= \frac{e^{-a}e^{ai}(-1-2i-(-1))}{8(1-(-1))} \\
&= \frac{e^{-a}e^{ai}(-2i)}{8(2)} \\
&= -\frac{e^{-a}e^{ai}i}{8} \\
&= -\frac{e^{-a}(\cos a + i\sin a)i}{8} \\
&= -\frac{e^{-a}(i\cos a + i^2\sin a)}{8} \\
&= -\frac{e^{-a}(i\cos a + (-1)\sin a)}{8} \\
&= \frac{e^{-a}(-i\cos a + \sin a)}{8} \\
&= \frac{-ie^{-a}\cos a + e^{-a}\sin a}{8} \\
&= \frac{e^{-a}\sin a}{8} - \frac{e^{-a}\cos a}{8}i.
\end{aligned}
$$

By the residue theorem from Section 77 applied to the region bounded by $[-R, R]$ and

C_R, we have

$$\begin{aligned}
\int_{-R}^{R} \frac{xe^{i(ax)}}{x^4+4}\,dx &= 2\pi i\left(\operatorname*{Res}_{z=-1+i} \frac{ze^{i(az)}}{z^4+1} + \operatorname*{Res}_{z=1+i} \frac{ze^{i(az)}}{z^4+1}\right) - \int_{C_R} \frac{ze^{i(az)}}{z^4+4}\,dz \\
&= 2\pi i\left(\left(\frac{e^{-a}\sin a}{8} + \frac{e^{-a}\cos a}{8}i\right) + \left(\frac{e^{-a}\sin a}{8} - \frac{e^{-a}\cos a}{8}i\right)\right) - \int_{C_R} \frac{ze^{i(az)}}{z^4+4}\,dz \\
&= 2\pi i\left(2\left(\frac{e^{-a}\sin a}{8}\right)\right) - \int_{C_R} \frac{ze^{i(az)}}{z^4+4}\,dz \\
&= \left(\frac{\pi}{2}e^{-a}\sin a\right)i - \int_{C_R} \frac{ze^{i(az)}}{z^4+4}\,dz.
\end{aligned}$$

Next, if we substitute $z = Re^{i\theta}$ and $dz = iRe^{i\theta}\,d\theta$ for all $0 < \theta < \pi$, we obtain

$$\begin{aligned}
\left|\int_{C_R} \frac{ze^{i(az)}}{z^4+4}\,dz\right| &= \left|\int_0^{\pi} \frac{Re^{i\theta}e^{i(aRe^{i\theta})}}{(Re^{i\theta})^4+4} iRe^{i\theta}\,d\theta\right| \\
&= \left|\int_0^{\pi} \frac{Re^{i\theta}e^{i(\cos\theta+i\sin\theta)}}{R^4e^{4i\theta}+4} iRe^{i\theta}\,d\theta\right| \\
&= \left|\int_0^{\pi} \frac{Re^{i\theta}e^{iaR\cos\theta+i^2aR\sin\theta}}{|R^4e^{4i\theta}+4|} iRe^{i\theta}\,d\theta\right| \\
&= \left|\int_0^{\pi} \frac{Re^{i\theta}e^{iaR\cos\theta-aR\sin\theta}}{|R^4e^{4i\theta}+4|} iRe^{i\theta}\,d\theta\right| \\
&= \left|\int_0^{\pi} \frac{Re^{i\theta}e^{iaR\cos\theta}e^{-aR\sin\theta}}{|R^4e^{4i\theta}+4|} iRe^{i\theta}\,d\theta\right| \\
&\le \int_0^{\pi} \frac{|Re^{i\theta}||e^{iaR\cos\theta}||e^{-aR\sin\theta}|}{||R^4e^{4i\theta}|-|4||} |iRe^{i\theta}|\,d\theta \\
&\le \int_0^{\pi} \frac{Re^{-aR\sin\theta}}{|R^4-4|} R\,d\theta \\
&= \frac{R^2e^{-aR\sin\theta}}{R^4-4} \int_0^{\pi} 1\,d\theta \\
&= \frac{R^2e^{-aR\sin\theta}}{R^4-4} \theta\Big|_0^{\pi} \\
&= \frac{R^2e^{-aR\sin\theta}}{R^4-4} (\pi-0) \\
&= \frac{\pi R^2e^{-aR\sin\theta}}{R^4-4},
\end{aligned}$$

which implies

$$
\begin{aligned}
0 &\le \left| \lim_{R\to\infty} \int_{C_R} \frac{ze^{i(az)}}{z^4+4}\, dz \right| \\
&= \lim_{R\to\infty} \left| \int_{C_R} \frac{ze^{i(az)}}{z^4+4}\, dz \right| \\
&\le \lim_{R\to\infty} \frac{\pi R^2 e^{-aR\sin\theta}}{R^4-4} \\
&= \lim_{R\to\infty} \frac{\pi R^2 e^{-aR\sin\theta}}{R^4-4} \frac{\frac{1}{R^2}}{\frac{1}{R^2}} \\
&= \lim_{R\to\infty} \frac{\pi e^{-aR\sin\theta}}{R^2-\frac{4}{R^2}} \\
&= \frac{\pi\cdot 0}{\infty - 0} \\
&= 0,
\end{aligned}
$$

from which we conclude

$$
\lim_{R\to\infty} \int_{C_R} \frac{ze^{i(az)}}{z^4+4}\, dz = 0.
$$

So we obtain

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{xe^{i(ax)}}{x^4+4}\, dx &= \lim_{R\to\infty} \int_{-R}^{R} \frac{xe^{i(ax)}}{x^4+4}\, dx \\
&= \lim_{R\to\infty} \left(\left(\frac{\pi}{2} e^{-a} \sin a\right) i - \int_{C_R} \frac{ze^{i(az)}}{z^4+4}\, dz \right) \\
&= \lim_{R\to\infty} \left(\frac{\pi}{2} e^{-a} \sin a\right) i - \lim_{R\to\infty} \int_{C_R} \frac{ze^{i(az)}}{z^4+4}\, dz \\
&= \left(\frac{\pi}{2} e^{-a} \sin a\right) i - 0 \\
&= 0 + \left(\frac{\pi}{2} e^{-a} \sin a\right) i.
\end{aligned}
$$

At the same time, Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{xe^{i(ax)}}{x^4+4}\, dx &= \int_{-\infty}^{\infty} \frac{x(\cos(ax) + i\sin(ax))}{x^4+4}\, dx \\
&= \int_{-\infty}^{\infty} \frac{x\cos(ax) + ix\sin(ax)}{x^4+4}\, dx \\
&= \int_{-\infty}^{\infty} \frac{x\cos(ax)}{x^4+4} + \frac{ix\sin(ax)}{x^4+4}\, dx \\
&= \int_{-\infty}^{\infty} \frac{x\cos(ax)}{x^4+4}\, dx + i\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^4+4}\, dx.
\end{aligned}
$$

Our two expressions of $\displaystyle\int_{-\infty}^{\infty} \frac{xe^{i(ax)}}{x^4+4}\, dx$ allow us to conclude

$$
\int_{-\infty}^{\infty} \frac{x\cos(ax)}{x^4+4}\, dx + i\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^4+4}\, dx = 0 + \left(\frac{\pi}{2} e^{-a} \sin a\right) i,
$$

from which we can equate the imaginary components to conclude

$$\int_{-\infty}^{\infty} \frac{x \sin(ax)}{x^4 + 4} \, dx = \boxed{\frac{\pi}{2} e^{-a} \sin a}$$

for all $a > 0$. □

88.5. $\displaystyle\int_{-\infty}^{\infty} \frac{x^3 \sin(ax)}{x^4 + 4} \, dx = \pi e^{-a} \cos a$ for all $a > 0$

The textbook made a typo of their final answer by writing $\pi e^{-a} \sin a$ instead of $\pi e^{-a} \cos a$. I entered this integral in WolframAlpha (https://www.wolframalpha.com/input?i=%5Cint_%7B-%5Cinfty%7D%5E%5Cinfty+%5Cfrac%7Bx%5E3%5Csin%28ax%29%7D%7Bx%5E4%2B4%7D+%5C%2C+dx); this input matches my boxed answer in my solution instead of the textbook's printed answer.

Solution. The isolated singularities of

$$\begin{aligned} \frac{z^3 e^{i(az)}}{z^4 + 4} &= \frac{z^3 e^{i(az)}}{(z^2 + 2i)(z^2 - 2i)} \\ &= \frac{z^3 e^{i(az)}}{(z - (-1 + i))(z - (-1 - i))(z - (1 + i))(z - (1 - i))} \end{aligned}$$

are $z = -1 \pm i$ and $z = 1 \pm i$, each of which are poles of order 1. But only $z = -1 + i$ and $z = 1 + i$ lie within the upper semicircle C_R with a sufficiently large radius $R > 0$. The residue of $\dfrac{z^3 e^{i(az)}}{z^4+4}$ at $z=-1+i$ is

$$
\begin{aligned}
\operatorname*{Res}_{z=-1+i} \frac{z^3 e^{i(az)}}{z^4+1} &= \left.\frac{z^3 e^{i(az)}}{(z-(-1-i))(z-(1+i))(z-(1-i))}\right|_{z=-1+i} \\
&= \frac{(-1+i)^3 e^{i(a(-1+i))}}{((-1+i)-(-1-i))((-1+i)-(1+i))((-1+i)-(1-i))} \\
&= \frac{((-1)^3+3(-1)^2 i+3(-1)i^2+i^3)e^{-ai+ai^2}}{(2i)(-2)(-2+2i)} \\
&= -\frac{(-1+3i-3i^2+i^3)e^{-ai+a(-1)}}{(2i)(-2)(2(-1+i))} \\
&= -\frac{(-1+3i+3-i)e^{-ai-a}}{-8(-i+i^2)} \\
&= -\frac{(2+2i)e^{-ai}e^{-a}}{8(-i+(-1))} \\
&= -\frac{2(1+i)e^{-ai}e^{-a}}{-8(1+i)} \\
&= \frac{2e^{-a}e^{-ai}(1+i)}{8(1+i)} \\
&= \frac{e^{-a}e^{-ai}}{4} \\
&= \frac{e^{-a}(\cos(-a)+i\sin(-a))}{4} \\
&= \frac{e^{-a}(\cos a - i\sin a)}{4} \\
&= \frac{e^{-a}\cos a - ie^{-a}\sin a}{4} \\
&= \frac{e^{-a}\cos a}{4} - \frac{e^{-a}\sin a}{4} i
\end{aligned}
$$

and the residue of $\frac{z^3 e^{i(az)}}{z^4+4}$ at $z = 1+i$ is

$$\begin{aligned}
\operatorname*{Res}_{z=1+i} \frac{z^3 e^{i(az)}}{z^4+4} &= \left.\frac{z^3 e^{i(az)}}{(z-(-1+i))(z-(-1-i))(z-(1-i))}\right|_{z=1+i} \\
&= \frac{(1+i)^3 e^{i(a(1+i))}}{((1+i)-(-1+i))((1+i)-(-1-i))((1+i)-(1-i))} \\
&= \frac{(1^3+3(1)^2 i+3(1)i^2+i^3)e^{ai+ai^2}}{(2)(2+2i)(2i)} \\
&= \frac{(1+3i+3i^2+i^3)e^{ai+a(-1)}}{(2)(2(1+i))(2i)} \\
&= \frac{(1+3i-3-i)e^{ai-a}}{8(1+i)} \\
&= \frac{(-2+2i)e^{ai}e^{-a}}{8(i+i^2)} \\
&= \frac{2(-1+i)e^{ai}e^{-a}}{8(i+(-1))} \\
&= \frac{e^{-a}e^{ai}(-1+i)}{4(-1+i)} \\
&= \frac{e^{-a}e^{ai}}{4} \\
&= \frac{e^{-a}(\cos a + i\sin a)}{4} \\
&= \frac{e^{-a}\cos a + ie^{-a}\sin a}{4} \\
&= \frac{e^{-a}\cos a}{4} + \frac{e^{-a}\sin a}{4} i.
\end{aligned}$$

By the residue theorem from Section 77 applied to the region bounded by $[-R, R]$ and C_R, we have

$$\begin{aligned}
\int_{-R}^{R} \frac{x^3 e^{i(ax)}}{x^4+4}\,dx &= 2\pi i \left(\operatorname*{Res}_{z=-1+i} \frac{z^3 e^{i(az)}}{z^4+1} + \operatorname*{Res}_{z=1+i} \frac{z^3 e^{i(az)}}{z^4+1}\right) - \int_{C_R} \frac{z^3 e^{i(az)}}{z^4+4}\,dz \\
&= 2\pi i \left(\left(\frac{e^{-a}\cos a}{4} - \frac{e^{-a}\sin a}{4} i\right) + \left(\frac{e^{-a}\cos a}{4} + \frac{e^{-a}\sin a}{4} i\right)\right) - \int_{C_R} \frac{z^3 e^{i(az)}}{z^4+4}\,dz \\
&= 2\pi i \left(2\left(\frac{e^{-a}\cos a}{4}\right)\right) - \int_{C_R} \frac{z^3 e^{i(az)}}{z^4+4}\,dz \\
&= (\pi e^{-a}\cos a)i - \int_{C_R} \frac{z^3 e^{i(az)}}{z^4+4}\,dz.
\end{aligned}$$

Next, if we substitute $z = Re^{i\theta}$ and $dz = iRe^{i\theta}\,d\theta$ for all $0 < \theta < \pi$, we obtain

$$\begin{aligned}
\left|\int_{C_R} \frac{z^3 e^{i(az)}}{z^4+4}\,dz\right| &= \left|\int_0^\pi \frac{(Re^{i\theta})^3 e^{i(aRe^{i\theta})}}{(Re^{i\theta})^4+4} iRe^{i\theta}\,d\theta\right| \\
&= \left|\int_0^\pi \frac{R^3 e^{3i\theta} e^{i(\cos\theta + i\sin\theta)}}{R^4 e^{4i\theta}+4} iRe^{i\theta}\,d\theta\right| \\
&= \left|\int_0^\pi \frac{R^3 e^{3i\theta} e^{iaR\cos\theta + i^2 aR\sin\theta}}{|R^4 e^{4i\theta}+4|} iRe^{i\theta}\,d\theta\right| \\
&= \left|\int_0^\pi \frac{R^3 e^{3i\theta} e^{iaR\cos\theta - aR\sin\theta}}{|R^4 e^{4i\theta}+4|} iRe^{i\theta}\,d\theta\right| \\
&= \left|\int_0^\pi \frac{R^3 e^{3i\theta} e^{iaR\cos\theta} e^{-aR\sin\theta}}{|R^4 e^{4i\theta}+4|} iRe^{i\theta}\,d\theta\right| \\
&\le \int_0^\pi \frac{|R^3 e^{3i\theta}||e^{iaR\cos\theta}||e^{-aR\sin\theta}|}{||R^4 e^{4i\theta}| - |4||} |iRe^{i\theta}|\,d\theta \\
&\le \int_0^\pi \frac{R^3 e^{-aR\sin\theta}}{|R^4-4|} R\,d\theta \\
&= \frac{R^4 e^{-aR\sin\theta}}{R^4-4} \int_0^\pi 1\,d\theta \\
&= \frac{R^4 e^{-aR\sin\theta}}{R^4-4} \theta\Big|_0^\pi \\
&= \frac{R^4 e^{-aR\sin\theta}}{R^4-4} (\pi - 0) \\
&= \frac{\pi R^4 e^{-aR\sin\theta}}{R^4-4},
\end{aligned}$$

which implies

$$\begin{aligned}
0 &\le \left|\lim_{R\to\infty} \int_{C_R} \frac{z e^{i(az)}}{z^4+4}\,dz\right| \\
&= \lim_{R\to\infty} \left|\int_{C_R} \frac{z e^{i(az)}}{z^4+4}\,dz\right| \\
&\le \lim_{R\to\infty} \frac{\pi R^4 e^{-aR\sin\theta}}{R^4-4} \\
&= \lim_{R\to\infty} \frac{\pi R^4 e^{-aR\sin\theta}}{R^4-4} \frac{\frac{1}{R^4}}{\frac{1}{R^4}} \\
&= \lim_{R\to\infty} \frac{\pi e^{-aR\sin\theta}}{1-\frac{4}{R^4}} \\
&= \frac{\pi \cdot 0}{1-0} \\
&= 0,
\end{aligned}$$

from which we conclude

$$\lim_{R\to\infty} \int_{C_R} \frac{z^3 e^{i(az)}}{z^4+4}\,dz = 0.$$

So we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^3 e^{i(ax)}}{x^4+4}\,dx &= \lim_{R\to\infty} \int_{-R}^{R} \frac{x^3 e^{i(ax)}}{x^4+4}\,dx \\
&= \lim_{R\to\infty} \left((\pi e^{-a}\cos a)i - \int_{C_R} \frac{z^3 e^{i(az)}}{z^4+4}\,dz \right) \\
&= \lim_{R\to\infty} (\pi e^{-a}\cos a)i - \lim_{R\to\infty} \int_{C_R} \frac{z e^{i(az)}}{z^4+4}\,dz \\
&= (\pi e^{-a}\cos a)i - 0 \\
&= 0 + (\pi e^{-a}\cos a)i.
\end{aligned}$$

At the same time, Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^3 e^{i(ax)}}{x^4+4}\,dx &= \int_{-\infty}^{\infty} \frac{x^3(\cos(ax) + i\sin(ax))}{x^4+4}\,dx \\
&= \int_{-\infty}^{\infty} \frac{x^3\cos(ax) + ix^3\sin(ax)}{x^4+4}\,dx \\
&= \int_{-\infty}^{\infty} \frac{x^3\cos(ax)}{x^4+4} + \frac{ix^3\sin(ax)}{x^4+4}\,dx \\
&= \int_{-\infty}^{\infty} \frac{x^3\cos(ax)}{x^4+4}\,dx + i\int_{-\infty}^{\infty} \frac{x^3\sin(ax)}{x^4+4}\,dx.
\end{aligned}$$

Our two expressions of $\int_{-\infty}^{\infty} \frac{x^3 e^{i(ax)}}{x^4+4}\,dx$ allow us to conclude

$$\int_{-\infty}^{\infty} \frac{x^3\cos(ax)}{x^4+4}\,dx + i\int_{-\infty}^{\infty} \frac{x^3\sin(ax)}{x^4+4}\,dx = 0 + (\pi e^{-a}\cos a)i,$$

from which we can equate the imaginary components to conclude

$$\int_{-\infty}^{\infty} \frac{x^3\sin(ax)}{x^4+4}\,dx = \boxed{\pi e^{-a}\cos a}$$

for all $a > 0$. □

Use residues to find the Cauchy principal values of the improper integrals in Exercises 8 through 11.

88.8. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx = -\frac{\pi}{e}\sin 2$

Solution. The isolated singularities of

$$\frac{e^{iz}}{z^2+4z+5} = \frac{e^{iz}}{(z-(-2-i))(z-(-2+i))}$$

are $z = -2 \pm i$, both of which are poles of order 1. But only $z = -2 + i$ lies within the upper semicircle C_R with a sufficiently large radius $R > 0$. The residue of $\frac{e^{iz}}{z^2+4z+5}$ at

$z = -2 + i$ is

$$
\begin{aligned}
\operatorname*{Res}_{z=-2+i} \frac{e^{iz}}{z^2+4z+5} &= \left.\frac{e^{iz}}{z-(2-i)}\right|_{z=-2+i} \\
&= \frac{e^{i(-2+i)}}{(-2+i)-(-2-i)} \\
&= \frac{e^{-2i+i^2}}{2i} \\
&= \frac{e^{-2i-1}}{2i} \\
&= \frac{e^{-2i}e^{-1}}{2i} \\
&= \frac{e^{-1}e^{-2i}}{2i}\frac{i}{i} \\
&= \frac{e^{-1}e^{-2i}i}{2i^2} \\
&= \frac{e^{-1}e^{-2i}i}{2(-1)} \\
&= -\frac{e^{-1}e^{-2i}i}{2} \\
&= -\frac{e^{-2i}i}{2e} \\
&= -\frac{(\cos(-2)+i\sin(-2))i}{2e} \\
&= -\frac{(\cos 2 - i\sin 2)i}{2e} \\
&= -\frac{i\cos 2 - i^2\sin 2}{2e} \\
&= -\frac{i\cos 2 - (-1)\sin 2}{2e} \\
&= -\frac{i\cos 2 + \sin 2}{2e} \\
&= -\frac{\sin 2}{2e} - \frac{\cos 2}{2e}i.
\end{aligned}
$$

By the residue theorem from Section 77 applied to the closed region bounded by $[-R, R]$

and C_R in the counterclockwise sense, we have

$$\begin{aligned}
\int_{-R}^{R} \frac{e^{ix}}{x^2+4x+5}\,dx &= 2\pi i \operatorname*{Res}_{z=bi} \frac{e^{iz}}{z^2+4z+5} - \int_{C_R} \frac{e^{iz}}{z^2+4z+5}\,dz \\
&= 2\pi i\left(-\frac{\sin 2}{2e} - \frac{\cos 2}{2e} i\right) - \int_{C_R} \frac{e^{iz}}{z^2+4z+5}\,dz \\
&= \left(-\frac{\pi}{e}\sin 2\right) i - \left(\frac{\pi}{e}\cos 2\right) i^2 - \int_{C_R} \frac{e^{iz}}{z^2+4z+5}\,dz \\
&= \left(-\frac{\pi}{e}\sin 2\right) i - \left(\frac{\pi}{e}\cos 2\right)(-1) - \int_{C_R} \frac{e^{iz}}{z^2+4z+5}\,dz \\
&= \frac{\pi}{e}\cos 2 + \left(-\frac{\pi}{e}\sin 2\right) i - \int_{C_R} \frac{e^{iz}}{z^2+4z+5}\,dz.
\end{aligned}$$

Next, if we substitute $z = Re^{i\theta}$ and $dz = iRe^{i\theta}\,d\theta$ for all $0 < \theta < \pi$, we obtain

$$\begin{aligned}
\left|\int_{C_R} \frac{e^{iz}}{z^2+4z+5}\,dz\right| &= \left|\int_0^{\pi} \frac{e^{i(Re^{i\theta})}}{(Re^{i\theta})^2+4(Re^{i\theta})+5}\,d\theta\right| \\
&= \left|\int_0^{\pi} \frac{e^{iR(\cos\theta+i\sin\theta)}}{R^2e^{2i\theta}+4Re^{i\theta}+5} iRe^{i\theta}\,d\theta\right| \\
&= \left|\int_0^{\pi} \frac{e^{iR\cos\theta+i^2R\sin\theta}}{R^2e^{2i\theta}+4Re^{i\theta}+5} iRe^{i\theta}\,d\theta\right| \\
&= \left|\int_0^{\pi} \frac{e^{iR\cos\theta-R\sin\theta}}{R^2e^{2i\theta}+4Re^{i\theta}+5} iRe^{i\theta}\,d\theta\right| \\
&= \left|\int_0^{\pi} \frac{e^{iR\cos\theta}e^{-R\sin\theta}}{R^2e^{2i\theta}+4Re^{i\theta}+5} iRe^{i\theta}\,d\theta\right| \\
&\le \int_0^{\pi} \frac{|e^{iR\cos\theta}e^{-R\sin\theta}|}{|R^2e^{2i\theta}+4Re^{i\theta}+5|}\,|iRe^{i\theta}\,d\theta| \\
&\le \int_0^{\pi} \frac{|e^{iR\cos\theta}e^{-R\sin\theta}|}{||R^2e^{2i\theta}|-|4Re^{i\theta}|-|5||}\,|iRe^{i\theta}\,d\theta| \\
&= \int_0^{\pi} \frac{e^{-R\sin\theta}}{|R^2-4R-5|} R\,d\theta \\
&= \frac{Re^{-R\sin\theta}}{R^2-4R-5}\int_0^{\pi} 1\,d\theta \\
&= \frac{Re^{-R\sin\theta}}{R^2-4R-5}\theta\Big|_0^{\pi} \\
&= \frac{Re^{-R\sin\theta}}{R^2-4R-5}(\pi-0) \\
&= \frac{\pi Re^{-R\sin\theta}}{R^2-4R-5},
\end{aligned}$$

which implies

$$\begin{aligned}
0 &\le \left|\lim_{R\to\infty}\int_{C_R}\frac{e^{iz}}{z^2+4z+5}\,dz\right| \\
&= \lim_{R\to\infty}\left|\int_{C_R}\frac{e^{iz}}{z^2+4z+5}\,dz\right| \\
&\le \lim_{R\to\infty}\frac{\pi Re^{-R\sin\theta}}{R^2-4R-5} \\
&= \lim_{R\to\infty}\frac{\pi Re^{-R\sin\theta}}{R^2-4R-5}\frac{\frac{1}{R}}{\frac{1}{R}} \\
&= \lim_{R\to\infty}\frac{\pi e^{-R\sin\theta}}{R-4-\frac{5}{R}} \\
&= \frac{\pi\cdot 0}{\infty-4-0} \\
&= 0,
\end{aligned}$$

from which we conclude

$$\lim_{R\to\infty}\int_{C_R}\frac{e^{iz}}{z^2+4z+5}\,dz = 0.$$

So we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+4x+5}\,dx &= \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{ix}}{x^2+4x+5}\,dx \\
&= \lim_{R\to\infty}\left(\frac{\pi}{e}\cos 2+\left(-\frac{\pi}{e}\sin 2\right)i-\int_{C_R}\frac{e^{iz}}{z^2+4z+5}\,dz\right) \\
&= \lim_{R\to\infty}\left(\frac{\pi}{e}\cos 2+\left(-\frac{\pi}{e}\sin 2\right)i\right)-\lim_{R\to\infty}\int_{C_R}\frac{e^{iz}}{z^2+4z+5}\,dz \\
&= \frac{\pi}{e}\cos 2+\left(-\frac{\pi}{e}\sin 2\right)i-0 \\
&= \frac{\pi}{e}\cos 2+\left(-\frac{\pi}{e}\sin 2\right)i.
\end{aligned}$$

At the same time, Euler's formula $e^{i\theta}=\cos\theta+i\sin\theta$ implies

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+4x+5}\,dx &= \int_{-\infty}^{\infty}\frac{\cos x+i\sin x}{x^2+4x+5}\,dx \\
&= \int_{-\infty}^{\infty}\frac{\cos x}{x^2+4x+5}+\frac{i\sin x}{x^2+4x+5}\,dx \\
&= \int_{-\infty}^{\infty}\frac{\cos x}{x^2+4x+5}\,dx+i\int_{-\infty}^{\infty}\frac{\sin x}{x^2+4x+5}\,dx.
\end{aligned}$$

Our two expressions of $\displaystyle\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+4x+5}\,dx$ allow us to conclude

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+4x+5}\,dx+i\int_{-\infty}^{\infty}\frac{\sin x}{x^2+4x+5}\,dx = \frac{\pi}{e}\cos 2+\left(-\frac{\pi}{e}\sin 2\right)i,$$

from which we can equate the imaginary components to conclude

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5}\, dx = \boxed{-\frac{\pi}{e} \sin 2}.$$

□

88.9. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 2x + 2}\, dx = \frac{\pi}{e}(\sin 1 + \cos 1)$

Solution. The isolated singularities of

$$\frac{ze^{iz}}{z^2 + 2z + 2} = \frac{ze^{iz}}{(z - (-1 - i))(z - (-1 + i))}$$

are $z = -1 \pm i$, both of which are poles of order 1. But only $z = -1 + i$ lies within the upper semicircle C_R with a sufficiently large radius $R > 0$. The residue of $\dfrac{ze^{iz}}{z^2 + 2z + 2}$ at

$z = -1 + i$ is

$$
\begin{aligned}
\operatorname*{Res}_{z=-1+i} \frac{ze^{iz}}{z^2+2z+2} &= \left.\frac{ze^{iz}}{z-(-1-i)}\right|_{z=-1+i} \\
&= \frac{(-1+i)e^{i(-1+i)}}{(-1+i)-(-1-i)} \\
&= \frac{(-1+i)e^{-i+i^2}}{2i} \\
&= \frac{(-1+i)e^{-i-1}}{2i} \\
&= \frac{(-1+i)e^{-i}e^{-1}}{2i} \\
&= \frac{(-1+i)e^{-1}e^{-i}}{2i}\frac{i}{i} \\
&= \frac{(-1+i)e^{-1}e^{-i}i}{2i^2} \\
&= \frac{(-1+i)e^{-1}e^{-i}i}{2(-1)} \\
&= -\frac{(-1+i)e^{-1}e^{-i}i}{2} \\
&= -\frac{(-1+i)e^{-i}i}{2e} \\
&= -\frac{(-1+i)(\cos(-1)+i\sin(-1))i}{2e} \\
&= -\frac{(-1+i)(\cos 1 - i\sin 1)i}{2e} \\
&= -\frac{(-i+i^2)(\cos 1 - i\sin 1)}{2e} \\
&= -\frac{(-i-1)(\cos 1 - i\sin 1)}{2e} \\
&= \frac{(1+i)(\cos 1 - i\sin 1)}{2e} \\
&= \frac{(\cos 1 - i\sin 1) + i(\cos 1 - i\sin 1)}{2e} \\
&= \frac{\cos 1 - i\sin 1 + i(\cos 1) - i^2(\sin 1))}{2e} \\
&= \frac{\cos 1 - i\sin 1 + i(\cos 1) - (-1)(\sin 1))}{2e} \\
&= \frac{(\cos 1 + \sin 1) + i(\cos 1 - \sin 1)}{2e} \\
&= \frac{\cos 1 + \sin 1}{2e} + \frac{\cos 1 - \sin 1}{2e}i.
\end{aligned}
$$

By the residue theorem from Section 77 applied to the closed region bounded by $[-R, R]$

and C_R in the counterclockwise sense, we have

$$\begin{aligned}
\int_{-R}^{R} \frac{xe^{ix}}{x^2+2x+2}\,dx &= 2\pi i \operatorname*{Res}_{z=bi} \frac{ze^{iz}}{z^2+2z+2} - \int_{C_R} \frac{ze^{iz}}{z^2+2z+2}\,dz \\
&= 2\pi i \left(\frac{\cos 1 + \sin 1}{2e} + \frac{\cos 1 - \sin 1}{2e} i \right) - \int_{C_R} \frac{ze^{iz}}{z^2+2z+2}\,dz \\
&= \frac{\pi}{e}(\cos 1 + \sin 1)i + \frac{\pi}{e}(\cos 1 - \sin 1)i^2 - \int_{C_R} \frac{ze^{iz}}{z^2+2z+2}\,dz \\
&= \frac{\pi}{e}(\cos 1 + \sin 1)i + \frac{\pi}{e}(\cos 1 - \sin 1)(-1) - \int_{C_R} \frac{ze^{iz}}{z^2+2z+2}\,dz \\
&= -\frac{\pi}{e}(\cos 1 - \sin 1) + \frac{\pi}{e}(\cos 1 + \sin 1)i - \int_{C_R} \frac{ze^{iz}}{z^2+2z+2}\,dz.
\end{aligned}$$

Next, if we substitute $z = Re^{i\theta}$ and $dz = iRe^{i\theta}\,d\theta$ for all $0 < \theta < \pi$, we obtain

$$\begin{aligned}
\left| \int_{C_R} \frac{ze^{iz}}{z^2+2z+2}\,dz \right| &= \left| \int_0^{\pi} \frac{Re^{i\theta} e^{i(Re^{i\theta})}}{(Re^{i\theta})^2 + 2(Re^{i\theta}) + 2}\,d\theta \right| \\
&= \left| \int_0^{\pi} \frac{Re^{i\theta} e^{iR(\cos\theta + i\sin\theta)}}{R^2e^{2i\theta} + 2Re^{i\theta} + 2} iRe^{i\theta}\,d\theta \right| \\
&= \left| \int_0^{\pi} \frac{Re^{i\theta} e^{iR\cos\theta + i^2 R\sin\theta}}{R^2e^{2i\theta} + 2Re^{i\theta} + 2} iRe^{i\theta}\,d\theta \right| \\
&= \left| \int_0^{\pi} \frac{Re^{i\theta} e^{iR\cos\theta - R\sin\theta}}{R^2e^{2i\theta} + 2Re^{i\theta} + 2} iRe^{i\theta}\,d\theta \right| \\
&= \left| \int_0^{\pi} \frac{Re^{i\theta} e^{iR\cos\theta} e^{-R\sin\theta}}{R^2e^{2i\theta} + 2Re^{i\theta} + 2} iRe^{i\theta}\,d\theta \right| \\
&\le \int_0^{\pi} \frac{|Re^{i\theta} e^{iR\cos\theta} e^{-R\sin\theta}|}{|R^2e^{2i\theta} + 2Re^{i\theta} + 2|} \,|iRe^{i\theta}\,d\theta| \\
&\le \int_0^{\pi} \frac{|Re^{i\theta} e^{iR\cos\theta} e^{-R\sin\theta}|}{||R^2e^{2i\theta}| - |2Re^{i\theta}| - |2||} \,|iRe^{i\theta}\,d\theta| \\
&= \int_0^{\pi} \frac{Re^{-R\sin\theta}}{|R^2 - 2R - 2|} R\,d\theta \\
&= \frac{R^2e^{-R\sin\theta}}{R^2 - 2R - 2} \int_0^{\pi} 1\,d\theta \\
&= \frac{R^2e^{-R\sin\theta}}{R^2 - 2R - 2} \theta\Big|_0^{\pi} \\
&= \frac{R^2e^{-R\sin\theta}}{R^2 - 2R - 2} (\pi - 0) \\
&= \frac{\pi R^2e^{-R\sin\theta}}{R^2 - 2R - 2},
\end{aligned}$$

which implies

$$\begin{aligned}
0 &\le \left| \lim_{R\to\infty} \int_{C_R} \frac{ze^{iz}}{z^2+2z+2}\, dz \right| \\
&= \lim_{R\to\infty} \left| \int_{C_R} \frac{ze^{iz}}{z^2+2z+2}\, dz \right| \\
&\le \lim_{R\to\infty} \frac{\pi R^2 e^{-R\sin\theta}}{R^2-2R-2} \\
&= \lim_{R\to\infty} \frac{\pi R^2 e^{-R\sin\theta}}{R^2-2R-2} \frac{\frac{1}{R^2}}{\frac{1}{R^2}} \\
&= \lim_{R\to\infty} \frac{\pi e^{-R\sin\theta}}{1-\frac{2}{R}-\frac{2}{R^2}} \\
&= \frac{\pi\cdot 0}{1-0-0} \\
&= 0,
\end{aligned}$$

from which we conclude

$$\lim_{R\to\infty} \int_{C_R} \frac{ze^{iz}}{z^2+2z+2}\, dz = 0.$$

So we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+2x+2}\, dx &= \lim_{R\to\infty} \int_{-R}^{R} \frac{xe^{ix}}{x^2+2x+2}\, dx \\
&= \lim_{R\to\infty} \left(-\frac{\pi}{e}(\cos 1 - \sin 1) + \frac{\pi}{e}(\cos 1 + \sin 1)i - \int_{C_R} \frac{ze^{iz}}{z^2+2z+2}\, dz \right) \\
&= \lim_{R\to\infty} \left(-\frac{\pi}{e}(\cos 1 - \sin 1) + \frac{\pi}{e}(\cos 1 + \sin 1)i \right) - \lim_{R\to\infty} \int_{C_R} \frac{ze^{iz}}{z^2+2z+2}\, dz \\
&= -\frac{\pi}{e}(\cos 1 - \sin 1) + \frac{\pi}{e}(\cos 1 + \sin 1)i - 0 \\
&= -\frac{\pi}{e}(\cos 1 - \sin 1) + \frac{\pi}{e}(\cos 1 + \sin 1)i.
\end{aligned}$$

At the same time, Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+2x+2}\, dx &= \int_{-\infty}^{\infty} \frac{x(\cos x + i\sin x)}{x^2+2x+2}\, dx \\
&= \int_{-\infty}^{\infty} \frac{x\cos x}{x^2+2x+2} + \frac{ix\sin x}{x^2+2x+2}\, dx \\
&= \int_{-\infty}^{\infty} \frac{x\cos x}{x^2+2x+2}\, dx + i \int_{-\infty}^{\infty} \frac{x\sin x}{x^2+2x+2}\, dx.
\end{aligned}$$

Our two expressions of $\displaystyle\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+2x+2}\, dx$ allow us to conclude

$$\int_{-\infty}^{\infty} \frac{x\cos x}{x^2+2x+2}\, dx + i \int_{-\infty}^{\infty} \frac{x\sin x}{x^2+2x+2}\, dx = -\frac{\pi}{e}(\cos 1 - \sin 1) + \frac{\pi}{e}(\cos 1 + \sin 1)i,$$

from which we can equate the imaginary components to conclude

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 4x + 5}\, dx = \boxed{\frac{\pi}{e}(\cos 1 + \sin 1)}.$$

□