MATH 165B discussion
University of California, Riverside

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Solutions to suggested homework problems from
Complex Variables and Applications, Ninth Edition by James Brown and Ruel Churchill
Homework 8: Section 94, Exercises 1(a)(b)(c), 2, 5

94.1. Let C denote the unit circle $|z| = 1$, described in the positive sense. Use the theorem in Section 93 to determine the value of $\Delta_C \arg f(z)$ when

(a) $f(z) = z^2$

Solution. The function $f(z) = z^2$ only contains a zero of order 2 at $z = 0$ in the domain interior to the unit circle $|z| = 1$, and so $f(z) = z^2$ is analytic in the domain. Furthermore, all analytic functions in a domain are also meromorphic functions in the same domain, which means $f(z) = z^2$ is also meromorphic in the domain. Furthermore, on the circle $|z| = 1$, which implies $z = e^{i\theta}$, we have

$$\begin{aligned} f(z) &= z^2 \\ &= (e^{i\theta})^2 \\ &= e^{i(2\theta)} \\ &= 1 \\ &\neq 0. \end{aligned}$$

By the theorem in Section 93, the winding number is

$$\begin{aligned} \frac{1}{2\pi}\Delta_C \arg f(z) &= Z - P \\ &= 2 - 0 \\ &= 2, \end{aligned}$$

where Z and P denote the number of zeros and poles inside C, respectively, both counting multiplicities. So we conclude

$$\begin{aligned} \Delta_C \arg f(z) &= 2\pi \frac{1}{2\pi}\Delta_C \arg f(z) \\ &= 2\pi(2) \\ &= \boxed{4\pi}. \end{aligned}$$

□

(b) $f(z) = \dfrac{1}{z^2}$

Solution. The function $f(z) = \frac{1}{z^2}$ only contains a pole of order 2 at $z = 0$ in the unit circle $|z| = 1$, and so $f(z) = \frac{1}{z^2}$ is meromorphic in the domain interior to the unit circle $|z| = 1$. Furthermore, on the circle $|z| = 1$, which implies $z = e^{i\theta}$, we have

$$\begin{aligned} f(z) &= \frac{1}{z^2} \\ &= \frac{1}{(e^{i\theta})^2} \\ &= \frac{1}{e^{i(2\theta)}} \\ &= e^{i(-2\theta)} \\ &\neq 0. \end{aligned}$$

By the theorem in Section 93, the winding number is

$$\begin{aligned} \frac{1}{2\pi}\Delta_C \arg f(z) &= Z - P \\ &= 0 - 2 \\ &= -2, \end{aligned}$$

where Z and P denote the number of zeros and poles inside C, respectively, both counting multiplicities. So we conclude

$$\begin{aligned} \Delta_C \arg f(z) &= 2\pi \frac{1}{2\pi}\Delta_C \arg f(z) \\ &= 2\pi(-2) \\ &= \boxed{-4\pi}. \end{aligned}$$

□

(c) $f(z) = \dfrac{(2z-1)^7}{z^3}$

Solution. The function $f(z) = \frac{(2z-1)^7}{z^3}$ contains a zero of order 7 at $z = \frac{1}{2}$ and a pole of order 3 at $z = 0$ in the unit circle $|z| = 1$, and so $f(z) = \frac{1}{z^2}$ is meromorphic in the domain interior to the unit circle $|z| = 1$. Furthermore, on the circle $|z| = 1$, which implies $z = e^{i\theta}$, we have

$$\begin{aligned} f(z) &= \frac{(2z-1)^7}{z^3} \\ &= \frac{(2e^{i\theta}-1)^7}{(e^{i\theta})^2} \\ &= \frac{(2e^{i\theta}-1)^7}{e^{i(2\theta)}} \\ &\neq 0. \end{aligned}$$

By the theorem in Section 93, the winding number is

$$\begin{aligned} \frac{1}{2\pi}\Delta_C \arg f(z) &= Z - P \\ &= 7 - 3 \\ &= 4, \end{aligned}$$

where Z and P denote the number of zeros and poles inside C, respectively, both counting multiplicities. So we conclude

$$\begin{aligned}\Delta_C \arg f(z) &= 2\pi \frac{1}{2\pi} \Delta_C \arg f(z) \\ &= 2\pi(4) \\ &= \boxed{8\pi}.\end{aligned}$$

□

94.2. Let f be a function which is analytic inside and on a positively oriented simple closed contour C, and suppose that $f(z)$ is never zero on C. Let the image of C under the transformation $w = f(z)$ be the closed contour Γ shown in Figure 114. Determine the value of $\Delta_C \arg f(z)$ from that figure; and, with the aid of the theorem in Section 93, determine the number of zeros, counting multiplicities, of f interior to C.

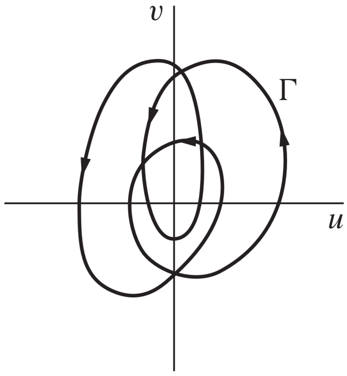

FIGURE 114

Proof. Fix any point w_0 on Γ. The net rotation of Γ about $w = 0$ as one travels along Γ from w_0 in the counterclockwise sense all the way back to w_0, we see that the net rotation about $w = 0$ is 3 revolutions; in other words, the winding number is 3. Furthermore, since f is assumed to be analytic inside C, there are no poles (namely, $P = 0$). So the theorem from Section 93 applies here, which allows us to conclude

$$\begin{aligned}Z &= Z - 0 \\ &= Z - P \\ &= \frac{1}{2\pi} \Delta_C \arg f(z) \\ &= \boxed{3}.\end{aligned}$$

In other words, f has three zeros, counting multiplicities, interior to C. □

94.5. Suppose that a function f is analytic inside and on a positively oriented simple closed contour C and that it has no zeros on C. Show that if f has n zeros z_k $(k = 1, 2, \ldots, n)$ inside C, where each z_k is of multiplicity m_k, then

$$\int_C \frac{z f'(z)}{f(z)}\, dz = 2\pi i \sum_{k=1}^{n} m_k z_k.$$

Solution. Since f is analytic and has n zeros z_k inside C, where each z_k is of multiplicity m_k, Theorem 1 of Section 82 asserts that there exists a function g, which is analytic and nonzero at each point z_k, that satisfies

$$f(z) = (z - z_k)^{m_k} g(z),$$

which also implies that first derivative of f can be written

$$\begin{aligned}
f'(z) &= \frac{d}{dz}(f(z)) \\
&= \frac{d}{dz}((z - z_k)^{m_k} g(z)) \\
&= \frac{d}{dz}((z - z_k)^{m_k})(g(z)) + ((z - z_k)^{m_k})\frac{d}{dz}(g(z)) \\
&= m_k(z - z_k)^{m_k - 1} g(z) + (z - z_k)^{m_k} g'(z) \\
&= (z - z_k)^{m_k - 1}(m_k g(z) + (z - z_k) g'(z)).
\end{aligned}$$

So we can write

$$\begin{aligned}
\frac{zf'(z)}{f(z)} &= \frac{z(z - z_k)^{m_k - 1}(m_k g(z) + (z - z_k) g'(z))}{(z - z_k)^{m_k} g(z)} \\
&= \frac{z(m_k g(z) + (z - z_k) g'(z))}{(z - z_k) g(z)} \\
&= \frac{z m_k g(z) + z(z - z_k) g'(z)}{(z - z_k) g(z)} \\
&= \frac{z m_k g(z)}{(z - z_k) g(z)} + \frac{z(z - z_k) g'(z)}{(z - z_k) g(z)} \\
&= \frac{z m_k}{z - z_k} + \frac{z g'(z)}{g(z)} \\
&= \frac{(z - z_k + z_k) m_k}{z - z_k} + \frac{z g'(z)}{g(z)} \\
&= \frac{(z - z_k) m_k + z_k m_k}{z - z_k} + \frac{z g'(z)}{g(z)} \\
&= \frac{(z - z_k) m_k}{z - z_k} + \frac{z_k m_k}{z - z_k} + \frac{z g'(z)}{g(z)} \\
&= m_k + \frac{m_k z_k}{z - z_k} + \frac{z g'(z)}{g(z)},
\end{aligned}$$

which implies that $\dfrac{zf'(z)}{f(z)}$ is meromorphic inside C with simple poles at each $z = z_k$ for $k = 1, 2, \ldots, n$, whose corresponding residues are

$$\operatorname*{Res}_{z = z_k} \frac{zf'(z)}{f(z)} = m_k z_k$$

for $k = 1, 2, \ldots, n$. Finally, by the residue theorem from Section 76, we conclude

$$\begin{aligned}
\int_C \frac{zf'(z)}{f(z)}\, dz &= 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z = z_k} \frac{zf'(z)}{f(z)} \\
&= 2\pi i \sum_{k=1}^{n} m_k z_k,
\end{aligned}$$

which is the desired result. □