Several Complex Variables

Notes by: Sara Lapan
Based on lectures given by: David Barrett

[0]These notes were typed during lecture and edited somewhat, so be aware that they are not error free. if you notice typos, feel free to email corrections to swlapan@umich.edu.

Lecture 1. January 7, 2008

Theorem 0.1. *Given $\Omega \subset \mathbb{C}$ open and $S \subset \Omega$ discrete (so S has no limit point in Ω). If $f : S \longrightarrow \mathbb{C}$ is an arbitrary function, then $\exists \tilde{f} : \Omega \longrightarrow \mathbb{C}$ holomorphic such that $\tilde{f} = f$.*

Remark 0.2. This theorem fails for some open subsets of $\mathbb{C}^n$ and holds for others. In one variable and for non-compact Riemann surfaces, this theorem holds. During this class we will answer the question: Exactly when does it hold?

An $\mathbb{R}$-differentiable function is one which is $\mathbb{R}$-linearly approximated. So it is helpful to know $\mathbb{R}$-linear algebra.
A holomorphic functions corresponds to $\mathbb{C}$-differentiable functions, which can be $\mathbb{C}$-linearly approximated. So it is helpful to know $\mathbb{C}$-linear algebra.
How do these relate?

Let V be a $\mathbb{C}$-vector space. One can view V as an $\mathbb{R}$-vector space with extra structure: $V \xrightarrow{J} V$ is given by $z \mapsto iz$, where J is $\mathbb{R}$-linear and $J^2 = -I$.
Conversely, any $\mathbb{R}$-vector space with such a map J becomes a $\mathbb{C}$-vector space: $(a + bi)z \equiv az + Jbz$

$T : V \longrightarrow W$ is $\mathbb{C}$-linear $\Leftrightarrow$ T is $\mathbb{R}$-linear and $TJ = JT \Leftrightarrow T$ is $\mathbb{R}$-linear and $T = -JTJ$

Definition 0.3. A map $T : V \longrightarrow W$ is **conjugate linear** (or anti-linear) if T is $\mathbb{R}$-linear and $T(\lambda z) = \overline{\lambda} Tz$. Equivalently, T is $\mathbb{R}$-linear and $TJ = -JT$

Proposition 0.4. *If T is $\mathbb{R}$-linear, then T admits a unique decomposition $T = T^{(1,0)} + T^{(0,1)}$, where $T^{(1,0)}$ is $\mathbb{C}$-linear and $T^{(0,1)}$ is conjugate linear.*

Proof. Uniqueness:

$$T = T^{(1,0)} + T^{(0,1)}$$
$$JTJ = -T^{(1,0)} + T^{(0,1)}$$
$$T^{(1,0)} = \frac{T - JTJ}{2} \text{ and } T^{(0,1)} = \frac{T + JTJ}{2}$$

Existence: Check that these work. □

Exercise 0.5. Let V and W be $\mathbb{C}$-vector spaces. Given an $\mathbb{R}$-linear map $T : V \longrightarrow W$, T is a $\mathbb{C}$-linear map $\Leftrightarrow$ the graph of T is a $\mathbb{C}$-subspace of $V \times W$.

Let V be a $\mathbb{C}$-vector space and $W \subset V$ an $\mathbb{R}$-linear subspace. Then W is a $\mathbb{C}$-linear subspace $\Leftrightarrow W = JW$. It is easy to check that $W \cap JW$ is the maximal $\mathbb{C}$-linear subspace of W.
Assume that V is finite dimensional:
$2\dim_{\mathbb{C}}(W \cap JW) = \dim_{\mathbb{R}}(W \cap JW) = \dim_{\mathbb{R}}(W) + \dim_{\mathbb{R}}(JW) - \dim_{\mathbb{R}} span(W \cup JW) \geq 2\dim_{\mathbb{R}}(W) - \dim_{\mathbb{R}}(V)$:

Proposition 0.6. $\frac{1}{2}\dim_{\mathbb{R}} W \geq \dim_{\mathbb{C}}(W \cap JW) \geq \dim_{\mathbb{R}}(W) - \dim_{\mathbb{C}}(V)$

Let $n = \dim_{\mathbb{C}} V$. Then:

$\dim_{\mathbb{R}} W$	$dim_{\mathbb{C}} W \cap JW$
0	0
1	0
$\vdots$	$\vdots$
n	$0, \ldots, [\frac{n}{2}]$ = greatest integer $\leq \frac{n}{2}$
$\vdots$	$\vdots$
$2n-2$	$n-2, n-1$
$2n-1$	$n-1$
$2n$	n

Exercise 0.7.

(1) Show that everything in the left column is possible. Hint: Consider $W = \mathbb{C}^k \times \mathbb{R}^l \times \{0\}^{2n-2k-l}$.
(2) Every W is isomorphic to that given in the hint. (so after a change in coordinates, every W can be characterized as given by the hint)

Definition 0.8. W is **totally real** if $W \cap JW = \{0\}$.

Corollary 0.9. *If* W *is totally real, then* $\dim_{\mathbb{R}} W = 2\dim_{\mathbb{C}} W \leq \dim_{\mathbb{C}} V$.

Definition 0.10. W is **maximally totally real** if W is totally real and $\dim_{\mathbb{R}} W = \dim_{\mathbb{C}} V$.

Warning: The term "generic" has two non-equivalent definitions. For this class the correct definition is:

Definition 0.11. W is **generic** if $\dim_{\mathbb{C}} W \cap JW = \max\{\dim_{\mathbb{R}} W - \dim_{\mathbb{C}} V, 0\}$.

The other common definition for generic (which is equivalent for large dimensions of W) is: W is **generic** if $\dim_{\mathbb{C}} W \cap JW = \dim_R W - \dim_{\mathbb{C}} V$.

Lecture 2. January 9, 2009

Let V be a $\mathbb{C}$-vector space, $W \subset V$ an $\mathbb{R}$-subspace and $W \cap JW$ a maximal $\mathbb{C}$-subspace of W.
Let $T : W \longrightarrow \mathbb{C}$. Does T extend to a $\mathbb{C}$-linear map $\tilde{T} : V \longrightarrow \mathbb{C}$? We would need T to be $\mathbb{R}$-linear and T to be $\mathbb{C}$-linear on $W \cap JW$. If so, define $\hat{T} : span_{\mathbb{R}} W \cup JW \longrightarrow \mathbb{C}$ by $v_1 + iv_2 \mapsto Tv_1 + iTv_2$.

Exercise 0.12. Show that $\hat{T}$ is well-defined and $\mathbb{C}$-linear.

Extend $\hat{T}$ to $\tilde{T}$ as usual. So $\tilde{T}$ exists $\Leftrightarrow$ T is $\mathbb{R}$-linear and T is $\mathbb{C}$-linear on $W \cap JW$.
$\tilde{T}$ is unique $\Leftrightarrow$ $span_{\mathbb{R}} W \cup JW = V$

$$\begin{aligned} \text{The Fundamental Theorem of Linear Algebra} &\Leftrightarrow \dim_{\mathbb{C}} W \cap JW = \dim_{\mathbb{R}} W - \dim_{\mathbb{C}} V \\ &\Leftrightarrow W \text{is generic (by the second definition).} \end{aligned}$$

Exercise 0.13. Let V and W both be $\mathbb{C}$-vector spaces. Which $\mathbb{R}$-linear maps $V \longrightarrow W$ send $\mathbb{C}$-subspaces to $\mathbb{C}$-subspaces? *Some maps that satisfy this are:* $\mathbb{C}$*-linear maps, conjugate-linear maps, surjective maps* $\mathbb{C} \longrightarrow \mathbb{C}$*, composition of the above. In fact, every such map can be created by combinations, in some way, of those maps.*

Let V be a $\mathbb{C}$-vector space and $Q : V \times V \longrightarrow \mathbb{R}$ be a bilinear, symmetric map.

Definition 0.14. Q is **$\mathbb{R}$-Hermitian** if $Q(Jv_1, Jv_2) = Q(v_1, v_2)$.

Definition 0.15. Q is **$\mathbb{R}$-anti-Hermitian** if $Q(Jv_1, Jv_2) = -Q(v_1, v_2)$. Equivalently, $Q(v_1, Jv_2) = -Q(Jv_1, v_2)$.

Proposition 0.16. *Q decomposes uniquely as* $Q = Q^{\textit{Herm.}} + Q^{\textit{anti-Herm.}}$.

Proof. $Q^{\text{Herm.}}(v_1, v_2) = \frac{Q(v_1,v_2)+Q(Jv_1,Jv_2)}{2}$ and $Q^{\text{anti-Herm.}}(v_1, v_2) = \frac{Q(v_1,v_2)-Q(Jv_1,Jv_2)}{2}$ □

Exercise 0.17. $Q((z_1, \dots, z_n)^t, (w_1, \dots, w_n)^t) = Re(\sum a_{jk} z_j \overline{w_k}) + Re(\sum b_{jk} z_j w_k)$ (where the first term is Hermitian and the second is anti-Hermitian, $a_{jk} = \overline{a_{kj}}$, and $b_{jk} = b_{kj}$).

If Q is a real, symmetric, bilinear form on V, the signature of Q is the triple (n_+, n_-, n_0) where:

$$\begin{aligned} n_+ &= \max\{\dim E \mid Q > 0 \text{ on } E\} \\ n_- &= \max\{\dim E \mid Q < 0 \text{ on } E\} \\ n_0 &= \dim\{v \in V \mid Q(v, w) = 0 \forall w \in V\}. \end{aligned}$$

Theorem 0.18. *We can choose an $\mathbb{R}$-basis such that Q can be replaced by:* $\begin{pmatrix} I_{n_+} & 0 & 0 \\ 0 & 0_{n_0} & 0 \\ 0 & 0 & -I_{n_-} \end{pmatrix}$.
Also, $n_+ + n_0 = \max\{\dim E \mid Q \geq 0 \text{ on } E\}$ *and* $n_- + n_0 = \max\{\dim E \mid Q \leq 0 \text{ on } E\}$. *This is also true if Q is $\mathbb{C}$-Hermitian.*

If V is a $\mathbb{C}$-vector space, how does the real signature of Q compare to the complex signature of $Q^{Herm.}$?
The sign (i.e. $>, < . \geq, \leq$) of the signature of Q on W, a subspace of V, gives the same sign for the complex signature of $Q^{\text{Herm.}}$ on $W \cap JW$.
Conclude: $n_+^{\mathbb{C}}(Q^{\text{Herm.}}) \geq n_+^{\mathbb{R}}(Q) - \dim_{\mathbb{C}} V$ and $n_-^{\mathbb{C}}(Q^{\text{Herm.}}) \geq n_-^{\mathbb{R}}(Q) - \dim_{\mathbb{C}} V$. The same inequalities hold if we add n_0 to the left-hand side of both inequalities.

Corollary 0.19. $Q^{Herm.} \geq 0 \Rightarrow n_-^{\mathbb{R}}(Q) \leq \dim_{\mathbb{C}} V$

Corollary 0.20. $n_+^{\mathbb{C}}(Q^{Herm.}) \leq n_+^{\mathbb{R}}(Q)$

Proof.

$$\begin{aligned} n_+^{\mathbb{C}}(Q^{\text{Herm.}}) &= \dim_{\mathbb{C}} V - n_0^{\mathbb{C}}(Q^{\text{Herm.}}) - n_-^{\mathbb{C}}(Q^{\text{Herm.}}) \\ &\leq \dim_{\mathbb{C}} V - (n_0^{\mathbb{R}}(Q) + n_-^{\mathbb{R}}(Q) - \dim_{\mathbb{C}} V) \\ &= \dim_{\mathbb{R}} V - n_0^{\mathbb{R}} V - n_-^{\mathbb{R}}(Q) \\ &= n_+^{R}(Q) \end{aligned}$$

□

Given an $\mathbb{R}$-Hermitian form Q on V, let $Q_{\mathbb{C}}(v_1, v_2) = Q(v_1, v_2) + iQ(v_1, Jv_2)$.
Then $Q_{\mathbb{C}}(Jv_1, v_2) = Q(v_1, Jv_2) - iQ(v_1, v_2) = -iQ_{\mathbb{C}}(v_1, v_2)$.
So $Q_{\mathbb{C}}$ is $\mathbb{C}$-linear in v_1 and conjugate-linear in $v_2 \Rightarrow Q_{\mathbb{C}}$ is $\mathbb{C}-$Hermitian.

Exercise 0.21. The real part of a $\mathbb{C}$-Hermitian form is $\mathbb{R}$-Hermitian.

Some people prefer to use $\mathbb{R}$-Hermitian and others prefer $\mathbb{C}$-Hermitian, so the above definition gives an easy way to go back and forth between the two terms.

Definition 0.22. Let $U \subset \mathbb{C}^n$ be open. Then $f : U \longrightarrow \mathbb{C}^k$ is **$\mathbb{C}$-differentiable** at $z_0 \in U$ if $\exists T : \mathbb{C}^n \longrightarrow \mathbb{C}^k$ that is $\mathbb{C}$-linear such that $\lim_{z \to z_0} \frac{||f(z) - f(z_0) - T(z - z_0)||}{||z - z_0||} = 0$.

Equivalently, f is $\mathbb{R}$-differentiable at z_0 and $T = f'(z_0) : \mathbb{C}^n \longrightarrow \mathbb{C}^k$ is $\mathbb{C}$-linear.
Equivalently, if $f = (f_1, \ldots, f_k)$, then each f_i is $\mathbb{C}$-differentiable at z_0. So we can focus on the case when $k = 1$.

Lecture 3. January 12, 2009

Let $z = x + iy, w = u + iv$, and $f = (f_1, \dots, f_k) \Rightarrow u_j = \frac{w_j + \overline{w_j}}{2}$ and $v_j = \frac{w_j - \overline{w_j}}{2i}$. Then:

$$\begin{aligned} f'(z_0)\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} &= f'(z_0)\begin{pmatrix} u_1 \\ v_1 \\ \vdots \\ u_n \\ v_n \end{pmatrix} \\ &= \frac{\partial f}{\partial x_1}(z_0)u_1 + \frac{\partial f}{\partial y_1}(z_0)v_1 + \dots \\ &= \frac{\frac{\partial f}{\partial x_1}(z_0) - i\frac{\partial f}{\partial y_1}(z_0)}{2} w_1 + \dots (\mathbb{C} - \text{linear}) \\ &+ \frac{\frac{\partial f}{\partial x_1}(z_0) + i\frac{\partial f}{\partial y_1}(z_0)}{2} \overline{w_1} + \dots (\mathbb{C} - \text{conj.-linear}) \end{aligned}$$

Note that:

$$\frac{\partial f}{\partial z_1}(z_0) = \frac{\frac{\partial f}{\partial x_1}(z_0) - i\frac{\partial f}{\partial y_1}(z_0)}{2} \text{ and } \frac{\partial f}{\partial \overline{z_1}}(z_0) = \frac{\frac{\partial f}{\partial x_1}(z_0) + i\frac{\partial f}{\partial y_1}(z_0)}{2}$$

$$df = \frac{\partial f}{\partial z_1}dz_1 + \dots + \frac{\partial f}{\partial \overline{z_1}}d\overline{z_1} + \dots = \partial f + \overline{\partial} f$$

$$\frac{\partial f}{\partial z_1}dz_1 = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial y_1}dy_1$$

So f is $\mathbb{C}$-differentiable at $z_0 \Leftrightarrow f$ is $\mathbb{R}$-differentiable at z_0 and $\frac{\partial f}{\partial \overline{z_j}} = 0 \forall j \Leftrightarrow f$ is $\mathbb{R}$-differentiable at z_0 and $\overline{\partial} f = 0$.

Definition 0.23. Let $U \subset \mathbb{C}^n$ be open. The following are equivalent:

(1) f is **holomorphic** on U
(2) f is continuously $\mathbb{C}$-differentiable on U
(3) f is $C^1_{\mathbb{R}}$ on U and $f'(z_0)J = Jf'(z_0)\ \forall z_0 \in U$
(4) f is $C^1_{\mathbb{R}}$ on U and $\frac{\partial f}{\partial \overline{z_j}} = 0\ \forall j$
(5) f is $C^1_{\mathbb{R}}$ on U and holomorphic in each variable separately

Proposition 0.24. *Holomorphic functions are closed under composition.*

Proof. Key linear algebra fact to use: $\mathbb{C}$ linear maps are closed under composition. □

Proposition 0.25. *Holomorphic functions are closed under addition.*

Proof. Key linear algebra fact to use: $\mathbb{C}$ linear maps are closed under addition. □

Proposition 0.26. *Scalar-valued holomorphic functions are closed under multiplication.*

Examples of holomorphic functions:

- Finite polynomials in $z_1, \dots, z_n$ (denoted by the vector z) with powers $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$ (denoted by the vector α). To simplify notation we use:

$$\sum c_{\alpha_1,\dots,\alpha_n} z_1^{\alpha_1} \dots z_n^{\alpha_n} = \sum c_\alpha z^\alpha \text{ and } \alpha \geq 0 \Rightarrow \alpha_j \geq 0 \forall j$$

- Power series $f(z) = \sum_{\alpha \geq 0} c_\alpha z^\alpha$ that coverges on $\{z \mid |z_j| \leq r_j\} \Rightarrow f$ is holomorphic on $\{z \mid |z_j| < r_j\}$.

Proof. On $\{z \mid |z_j| < r_j\}$, $|c_\alpha| r^\alpha \leq M$ for some M.
$|z_j| \leq \lambda_j r_j, \lambda_j < 1 \Rightarrow \sum |c_\alpha z^\alpha| \leq M \sum \lambda^\alpha = M \prod_{j=1}^n \frac{1}{1-\lambda_j} < \infty$. So f converges absolutely on $\{z \mid |z_j| \leq r_j\}$.
Now show holomorphic: Define: ${\frac{\partial}{\partial z}}^\beta = (\frac{\partial}{\partial z_1})^{\beta_1} \dots (\frac{\partial}{\partial z_n})^{\beta_n}$
$\sum_\alpha |(\frac{\partial}{\partial z})^\beta c_\alpha z^\alpha| = \sum \gamma_{\alpha,\beta} |c_\alpha z^{\alpha-\beta}|$, where $\gamma_{\alpha,\beta} = \prod \frac{(\alpha_j)!}{(\alpha_j - \beta_j)!}$ or 0. Then

$$\begin{aligned}\sum_\alpha |(\frac{\partial}{\partial z})^\beta c_\alpha z^\alpha| &\leq \sum \gamma_{\alpha,\beta} |c_\alpha| r^{\alpha-\beta} \lambda^{\alpha-\beta} \\ &\leq \frac{M}{r^\beta} \sum \gamma_{\alpha,\beta} \lambda^{\alpha-\beta} \\ &= \frac{M}{r^\beta} \sum (\frac{\partial}{\partial \lambda})^\beta \lambda^\alpha \\ &= \tilde{M} \prod_{j=1}^n \frac{1}{1-\lambda_j}^{1+\beta_j}\end{aligned}$$

So $\overline{\partial}(\sum c_\alpha z^\alpha) = \sum \overline{\partial}(c_\alpha z^\alpha) = 0$ and all derivatives converge uniformly on compact subsets of $\{z \mid |z_j| < r_j\}$. □

Remark 0.27. This result generalizes for $\sum b_\alpha (z-c)^\alpha$ on $\{z \mid |z_j - c_j| < r_j\}$.

Lecture 4. January 14, 2009

Suppose that f is holomorphic on a neighborhood of $\{z \mid |z - c_j| \leq r_j\}$. Then

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|\varphi_1 - c_1| = r_1} \frac{f(\varphi_1, z_2, \dots, z_n)}{\varphi_1 - z_1} d\varphi_1 \\ &= (\frac{1}{2\pi i})^2 \int_{|\varphi_2 - c_2| = r_2} \int_{|\varphi_1 - c_1| = r_1} \frac{f(\varphi_1, \varphi_2, z_3, \dots, z_n)}{(\varphi_1 - z_1)(\varphi_2 - z_2)} d\varphi_1 \wedge d\varphi_2 \\ &= \dots (n - times) \\ &= (\frac{1}{2\pi i})^n \int_{|\varphi_1 - c_1| < r_1} \dots \int_{|\varphi_n - c_n| < r_n} \frac{f(\varphi_1, \dots \varphi_n)}{(\varphi_1 - z_1) \dots (\varphi_n - z_n)} d\varphi_1 \wedge \dots \wedge d\varphi_n \end{aligned}$$

Note that: $\varphi_j = c_j + r_j e^{i\theta_j}$. Then $d\varphi_1 \wedge \dots \wedge d\varphi_n = i^n r_1 \dots r_n e^{i(\theta_1 \dots \theta_n)} d\theta_1 \wedge \dots \wedge \theta_n$.

$$\begin{aligned} \frac{1}{\varphi_j - z_j} &= \frac{1}{\varphi_j - c_j} \frac{1}{1 - \frac{z_j - c_j}{\varphi_j - c_j}} \\ &= \sum_{\alpha \geq 0} \frac{(z_j - c_j)^{\alpha_j}}{(\varphi_j - c_j)^{\alpha_j + 1}} \end{aligned}$$

Using this in the above formula for f:

$$f(z) = \sum_{\alpha \geq 0} \left(\frac{1}{(2\pi i)^n} \int \dots \int \frac{f(\varphi) d\varphi_1 \wedge \dots \wedge d\varphi_n}{(\varphi_1 - c_1)^{\alpha_1 + 1} \dots (\varphi_n - c_n)^{\alpha_{n+1}} (z-c)^\alpha} \right)$$

So f is holomorphic $\Leftrightarrow$ f is locally defined by power series. In $\mathbb{C}^1$, the set of convergences for $\sum b_n z^n$ is a disk (together with some boundary points) or $\mathbb{C}$ or $\{0\}$.

Exercise 0.28. The interior for $\{z \mid \sum b_\alpha z^\alpha$ converges $\}$ is a union of open polydisks centered at 0. More to be said later.

Remark 0.29. The above computations work for a functions that is continous and holomorphic in each variable z_j. Therefore such functions are guaranteed to be holomorphic.

Goursat's Theorem in one variable says that as long as f is $\mathbb{C}$-differentiable at each point (we do not need to assume that f' is continous), then f is holomorphic. Therefore by this result and the one above we conclude that if f is $\mathbb{C}$-differentiable at each point (for multiple variables), then f is holomorphic.

Theorem 0.30 (Hartog's). *If f is holomorphic in each z_j, then f is holomorphic.*

Hard: Omit for now. □

This clearly does not hold in real calculus.
Suppose $f(z) = \sum b_\alpha (z-c)^\alpha$. We are working on a polydisk and we want a formula for the center of the disk:
$\frac{\partial^\alpha f}{\partial z^\alpha}(c) = \alpha_1! \dots \alpha_n! b_\alpha \equiv \alpha! b_\alpha$.

Corollary 0.31. *Suppose that f is holomorphic on Ω, a connected open set, and $\frac{\partial^\alpha f}{\partial z^\alpha}(c) = 0 \forall \alpha$, then $f \equiv 0$ on Ω.*

Corollary 0.32. *Suppose that $f : \Omega \longrightarrow \mathbb{C}$ is holomorphic and Ω is a connected, open set. Also, suppose that $|f|$ has an interior maximum, then f is constant on Ω.*

Proof. Use slices to break f down into one variable and then use the one-dimensional result. □

Corollary 0.33. *Suppose that $f : \Omega \longrightarrow \mathbb{C}$ is holomorphic and Ω is a connected, open set, then $f(\Omega)$ is either one point or open.*

Proof. Use slices to break f down into one variable and use the open mapping theorem from one dimension. □

Example 0.34. Let $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be given by $(z, w) \mapsto (z, 0)$. Then $f(\mathbb{C}^2) = \mathbb{C} \times \{0\}$.

From this result you might want to ammend the above corollary to say that when mapping into $\mathbb{C}^n$, the image is small. But this does not work:

Example 0.35. Let $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be given by $(z, w) \mapsto (zw, w)$. Then $f(\mathbb{C}^2) = \{(z, w) \mid w \neq 0\} \cup \{(0, 0)\}$.

Lecture 5. January 16, 2009

If f is holomorphic in a neighborhoood of $\{z \mid |z_j - c_j| \leq c_j\}$, then $f(z) = \sum_{\alpha > 0} b_\alpha (z-c)^\alpha$ where

$$\begin{aligned} b_\alpha &= \frac{\frac{\partial^\alpha f}{\partial z^\alpha}(c)}{\alpha!} \\ &= (\frac{1}{2\pi i})^n \int \dots \int \frac{f(\varphi_1, \dots, \varphi_n) d\varphi_1 \wedge \dots \wedge d\varphi_n}{(\varphi - c)^{\alpha+1}} \\ &= (\frac{1}{2\pi})^n \int_0^{2\pi} \dots \int_0^{2\pi} \frac{f(c + re^{i\theta})}{r^\alpha e^{i\alpha\theta}} d\theta_1 \wedge \dots \wedge d\theta_n \end{aligned}$$

Note that:$re^{i\theta} = (r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})$

Corollary 0.36 (Cauchy Estimates). *If f is holomorphic on Ω and $|f| \leq M$ on Ω and $\{z | |z_j - c_j| \leq r\} \subseteq \Omega$, then $|\frac{\partial^\alpha f}{\partial z^\alpha}(c)| \leq \frac{M\alpha!}{r^{|\alpha|}}$, where $|\alpha| = \alpha_1 + \dots + \alpha_n$.*

Corollary 0.37 (Weierstrass Convergence Theorem). *$f_j \longrightarrow f$ almost uniformly (so uniformly on each compact subset of Ω) on Ω where f_j are holomorphic, then f is holomorphic on Ω.*

Proof. The convergence of the derviatives are also almost unform so that $\frac{\partial f_j}{\partial \overline{z_k}} \longrightarrow \frac{\partial f}{\partial \overline{z_k}} = 0$ □

Definition 0.38. A set of functions $\{f_j\}$ is equibounded if for any compact subset $K \subset \Omega, \exists M_k$ such that $|f_j| \leq M_k$ on $K \forall j$.

Corollary 0.39 (Montel's Theorem)**.** *If $\{f_j\}$ are holomorphic on Ω and equibounded oneach compact subset, then there exists a subsequence that converges almost uniformly on Ω.*

Proof. This follows as in the one-dmensional case from Cauchy estimates and Arzela-Ascoli theorem. □

Remark 0.40. So far all of the results and methods of proof have been like those in the one variable case, but the next result is a very surprising change.

$$H_1 = \{(z, w) \in \mathbb{C}^2 \mid 1 - \epsilon < |z| < 1, |w| < 1\}$$

$$H_2 = \{(z, w) \in \mathbb{C}^2 \mid |z| < 1, |w| < \epsilon\}$$

Let $H = H_1 \cup H_2$. This space has many names including: tomato can, cake pan, top hat, Hartegs configuration.
Suppose that f is holomorphic on H. On H_2 we have:

$$f(z, w) = \frac{1}{2\pi i} \int_{|\varphi|=r} \frac{f(\varphi, w)}{\varphi - z} d\varphi, \text{ where } |z| < r < 1$$

Define $\tilde{f}(z, w)$ on $\Delta \times \Delta$, where Δ is the unit disk, by:

$$\tilde{f}(z, w) = \frac{1}{2\pi i} \int_{|\varphi|=r} \frac{f(\varphi, w)}{\varphi - z} d\varphi, \text{ where } max\{1 - \epsilon, |z|\} < r < 1, \text{ then}$$

- $\tilde{f}$ is well-defined on $\Delta \times \Delta$
- $\tilde{f}$ is holomorphic on $\Delta \times \Delta$ (since we can switch the derivative and integral and we know that the functions being integrated is holomorhic)

$\tilde{f} = f$ on H_2, then $\tilde{f} = f$ on H (unique continuation principle). Therefore f extends to a holomorphic function on $\Delta \times \Delta$.

Remark 0.41. There is no similar result to this in $\mathbb{C}$. For instance, the functions $f(z) = \frac{1}{z - z_0}$ is holomorphic on an open set not containing z_0 but not in a larger open set that does contain z_0.

Recall the theory of isolated singularieties in $\mathbb{C}$:

- Removable singularities
- Poles
- Essential singularities

Isolated singularities in $\mathbb{C}^2$ are all removable - "We generalize one variable theory with an ax."

Proof. Let f be holomorphic on $\Omega \setminus \{z_0\}$. After rescaling and translating, we can get a Hartog's configuration around z_0 (but not containing). Let P be a polydisk and H be the Hartog's configuration. Get $\tilde{f}$ holomorphic on P with $\tilde{f} = f$ on H. Extend f to Ω. □

Lecture 6. January 21, 2009

$\Delta \equiv \{z \in \mathbb{C} \mid |z| < 1\}$
Consider $\zeta : \overline{\Delta} \longrightarrow \mathbb{C}^{n-1}$ continuous and let $0 < \epsilon < 1$

$$H_1 = \{(z, w_1, \ldots, w_{n-1}) \in \mathbb{C}^n \mid 1 - \epsilon < |z| < 1, ||w|| < 1\}$$
$$H_2^\zeta = \{(z, w_1, \ldots, w_{n-1})\mathbb{C}^n \mid |z| < 1, ||\zeta(z) - w|| < \epsilon\|$$
$$H^\zeta = H_1 \cup H_2^\zeta$$

Theorem 0.42 (Hartogs:1906)**.** *$\zeta \equiv 0, f$ holomorphic on $H \Rightarrow f$ extends to holomorphic functions on Δ^n.*

This roughly looks like a goal post or an H.

Remark 0.43. There is no automatic extension from Δ^n to fixed larger open sets.

Theorem 0.44 (Chirka: 1990s)**.** *When $n = 2$, $\zeta(\overline{\Delta} \subset \Delta, f$ holomorhpic on $H^\zeta \Rightarrow f$ extends to holomorphic functions on $\Delta^2 \cup H^\zeta$.*

Remark 0.45 (Rosay: 1998). This does not work when $n > 1$.

Theorem 0.46 (B.-Bharali: 2004)**.** *When $n = 2$, suppose that ζ has this form: $\zeta(re^{i\theta}) = \sum_{n\geq 0} \zeta_n(r)e^{in\theta}$, where ζ_n are functions in one variable, and the sum is finite or uniformly convergent and continuous (so $\zeta_n(0) = 0 \forall n > 0$) $\Rightarrow \exists \tilde{f}$ holomorphic on Δ^2 such that $\tilde{f} = f$ on H_1.*

Theorem 0.47 (B.-Bhardi: 2004)**.** *If $n = 2, \zeta(re^{i\theta}) = \sum_{n \in \mathbb{Z}, finite} \zeta_n(r)e^{in\theta}$ such that $\sum \frac{|\zeta_n(r)|}{r^n} < 1, \exists \tilde{f}$ holomorphic on Δ^2 such that $\tilde{f} = f$ on H_1.*

$\Omega \subset \mathbb{C}$ is a bounded open set. $b\Omega$ is C^1 (the boundary of Ω) and $f \in C^1(\overline{\Omega})$
By Stoke's/Green's Theorem:

$$\begin{aligned}
\int_{b\Omega} \frac{f(\zeta)d\zeta}{\zeta - z} - \int_{|\zeta - z| = \epsilon} \frac{f(\zeta)d\zeta}{\zeta - z} &= \int_{\zeta\in\Omega, |\zeta - z| > \epsilon} d_\zeta \left(\frac{f(\zeta)}{\zeta - z} d\zeta \right) \\
&= \int d \left(\frac{f(\zeta)}{\zeta - z} \wedge d\zeta \right. \\
&= \int \left(\frac{\partial}{\partial \zeta} (\frac{f(\zeta)}{\zeta - z} d\zeta + \frac{\partial}{\partial \overline{\zeta}} (\frac{f(\zeta)}{\zeta - z} d\overline{\zeta} \right) \wedge d\zeta \\
&= \int \frac{1}{\zeta - z} \frac{\partial}{\partial \overline{\zeta}} f(\zeta) d\overline{\zeta} \wedge d\zeta \\
&= \int \frac{\overline{\partial} f \wedge d\zeta}{\zeta - z} \\
&= 2i \int_{\zeta\in\Omega, |\zeta - z| > \epsilon} \frac{\frac{\partial f}{\partial \overline{\zeta}}}{\zeta - z} dA_\zeta
\end{aligned}$$

Side work:

$$\begin{aligned}
d\overline{\zeta} \wedge d\zeta &= (dx - idy) \wedge (dx + idy) \\
&= 2idx \wedge dy
\end{aligned}$$

Let $\epsilon \to 0$, note that:

$$\int \frac{f(\zeta)d\zeta}{\zeta - z} = \int_0^{2\pi} f(z + \epsilon e^{i\theta} d\theta \xrightarrow{\epsilon \to 0} 2\pi i f(z)$$

In the end, we get the Cauchy-Green Theorem:

$$\int_{\zeta\in b\Omega}\frac{f(\zeta)d\zeta}{\zeta-z}=2\pi i f(z)+2i\int_{\Omega}\frac{\frac{\partial f}{\partial\bar{\zeta}}}{\zeta-z}dA_\zeta$$

Given Ω as above and g continuous on $\overline{\Omega}$. We want to solve $\frac{\partial f}{\partial\bar{z}}=g$ on Ω. Suppose we can solve with $f\equiv 0$ on $b\Omega$. Then:

$$f(z)=-\frac{1}{\pi}\int_{\zeta\in\Omega}\frac{g(\zeta)}{\zeta-z}dA$$

This is called the **Cauchy Transform of g:** $\mathcal{C}_\Omega(g)$. Check that this integral is defined in the Lebesgue sense.

Remark 0.48. It turns out that this assumption is too optimistic so that it is not always possible to find a solution on the boundary with $f\equiv 0$, but it turns out that the formula we get from thinking about it is helpful.

Lecture 7. January 23, 2009

Theorem 0.49. *$g\in C(\overline{\Omega})\cap C^k(\Omega)\Rightarrow \mathcal{C}_\Omega(g)\in C^k(\Omega)$ and $\frac{\partial}{\partial\bar{z}}(\mathcal{C}_\Omega g)=g$.*

Proof. Fix $z_0\in\Omega$ and pick $\chi\in C_0^\infty(\Omega)$ (so χ has compact support in Ω, i.e. $X\equiv 0$ outside some compact subset of Ω) such that $\chi\equiv 1$ near z_0.

$$\begin{aligned}\mathcal{C}_\Omega g(z)&=\mathcal{C}_\Omega(\chi g)(z)+\mathcal{C}((1-\chi)g)(z)\\&=\frac{-1}{\pi}\int_{\Omega-z}\frac{(\chi g)(\zeta+z)}{\zeta}dA_\zeta-\frac{1}{\pi}\int_\Omega\frac{((1-\chi)(\zeta)}{\zeta-z}dA_\zeta\end{aligned}$$

"extended by zero" and "$\zeta-z$ is our new ζ"

Claim: We can differentiate under the integrals k-times.
Exercise: Check this carefully for (atleast) $k=1$. Use the Lebesgue dominated convergence theorem.
In particular,

$$\frac{\partial}{\partial\bar{z}}(\mathcal{C}_\Omega g)(z)=-\frac{1}{\pi}\int\frac{\frac{\partial(chig)}{\partial\bar{\zeta}}(\zeta+z)}{\zeta}dA_\zeta \tag{1}$$

$$=-\frac{1}{\pi}\int_\Omega\frac{\frac{\partial(\chi g}{\partial\bar{\zeta}}(\zeta)}{\zeta-z}dA_\zeta \tag{2}$$

$$=(\chi g)(z) \tag{3}$$

$$=g(z) \tag{4}$$

Where between steps 2 and 3 we use the Cauchy-Green equation and in the last step we are looking at z near z_0. ☐

Again, $\Omega\subset\mathbb{C}$ is a bounded open subset and $b\Omega C^1$.
Question: Can we solve: $\frac{\partial f}{\partial\bar{z}}=g$ on Ω and $f\equiv 0$ on $b\Omega$?

If so, we get:

$$\begin{aligned}
0 &= \int_{b\Omega} f dz \\
&= \int_{\Omega} d(f dz) \\
&= \int (\partial f + \overline{\partial} f) \wedge dz \\
&= \int_{\Omega} (\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z}) \wedge dz \\
&= \int_{\Omega} \frac{\partial f}{\partial \overline{z}} d\overline{z} \wedge dz \\
&= 2i \int_{\Omega} \frac{\partial f}{\partial \overline{z}} dA \\
&= 2i \int_{\Omega} g dA
\end{aligned}$$

So most of the time we cannot solve the question. Note: we also need $\int_\Omega fhdA = 0$ for h holomorphic near $\overline{\Omega}$.
Suppose $g \in C_0^1(\mathbb{C})$. $\mathcal{C}_{r\Delta} g$ independent of r for r large, and we get $\mathcal{C}_{\mathbb{C}} g$ on $\mathbb{C}$. $\frac{\partial}{\partial \overline{z}}(\mathcal{C}_{\mathbb{C}} g) = g$. If $\mathcal{C}_{\mathbb{C}} g$ has compact suppose then we could repeat the above argument to rpove that $\int_{\mathbb{C}} gdA = 0$.
We now want to move to $\mathbb{C}^n$:
Cauchy Riemann equations are: $\overline{\partial} f = 0$ where $\overline{\partial} f = \sum \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j$. Equivalently, $\frac{\partial f}{\partial \overline{z}_j} = 0 \forall j$.

Inhomogenous Cauchy-Riemann equations are: $\overline{\partial} f = g = \sum g_j(z) d\overline{z}_j$. Equivalently, $\frac{\partial f}{\partial \overline{z}_j} = g_j \forall j$. This expression is called a (0,1)-form.

In the one variable setting we can always solve this, but not in the several variable setting. For there to be a solution, we need:

$$\frac{\partial g_j}{\partial \overline{z}_k} = \frac{\partial g_k}{\partial \overline{z_j}}$$

Differential Forms Explanation:
Using the Inhomogenous equation: $g = \sum g_j(z) d\overline{z}_j$,

$$dg = \sum dg_j \wedge d\overline{z}_j \tag{5}$$
$$= \sum \partial g_j \wedge d\overline{z}_j + \sum \overline{\partial} g_j \wedge d\overline{z}_j \tag{6}$$
$$= \partial g + \overline{\partial} g \tag{7}$$
$$= (1,1) - \text{form } + (0,2) - \text{form} \tag{8}$$
$$\partial g = \sum \frac{\partial g_j}{\partial z_k} dz_k \wedge d\overline{z}_j \tag{9}$$
$$\overline{\partial} g = \sum \frac{\partial g_j}{\partial \overline{z_k}} d\overline{z}_k \wedge d\overline{z}_j \tag{10}$$

Step 1 to 2, we used the fact that $dg_j = \partial d_j + \overline{\partial} d_j$.
If h is a (1,0)-form, then $dh = \partial h + \overline{\partial} h$ is the sum of a (2,0)-form add a (1,1)-form.

From exterior algebra, we know that $d^2 f = 0$.

$$\begin{aligned} 0 &= d^2 f \\ &= d(\partial + \overline{\partial}) f \\ &= (\partial + \overline{\partial})(\partial + \overline{\partial}) f \\ &= \partial^2 f + \partial\overline{\partial} f + \overline{\partial}\partial f + \overline{\partial}^2 f \\ &= \text{(2,0)-form, two (1,1)-forms, and a (0,2)-form} \end{aligned}$$

So, $\partial^2 f = 0, \overline{\partial}^2 f = 0, \partial\overline{\partial} f = -\overline{\partial}\partial f$.

Thus to solve $\overline{\partial} f = g$, we need:

$$\overline{\partial} g = 0 \Leftrightarrow \frac{\partial g_j}{\partial \overline{z}_k} = \frac{\partial g_k}{\partial \overline{z}_j}$$

Lecture 8. January 26, 2009

Definition 0.50. The **support** of f is the closure of the non-zero set of f in the domain of f (i.e. $\overline{\{z | f(z) \neq 0\}}$).

We are given $g = \sum g_j(z) d\overline{z}_j$ on $\mathbb{C}^n$, $\overline{\partial} g = 0$ (i.e. $\frac{\partial g_j}{\partial \overline{z}_j} = \frac{\partial g_k}{\partial \overline{z}_j}$) and g has compact suppport.
To solve $\overline{\partial} f = g$ (i.e. $\frac{\partial f}{\partial \overline{z}_j} = g_j$).
Let $\mathcal{C}_g(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{g_1(\zeta, z_2, \ldots, z_n}{\zeta - z_1} dA_\zeta$.
Last time we showed that: $\frac{\partial}{\partial \overline{z}_1} \mathcal{C}_g = g_1$.
This tells us that all we need to know is g_1, we do not even need the other g_k's.
Claim:$\overline{\partial} \mathcal{C} g = g$ (i.e. $\frac{\partial}{\partial \overline{z}_k} \mathcal{C}_g = g_k$).

Proof. $\frac{\partial}{\partial \overline{z}_k} \mathcal{C}_g(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\frac{\partial g_1}{\partial \overline{z}_k}(\zeta, z_2, \ldots, z_n}{\zeta - z_1} dA_\zeta = g_k(z)$ by the Cauchy-Green equation. □

Claim:$\mathcal{C}_g$ has compact support, when $n \geq 2$ (recall that this is false when $n = 1$).

Proof. Outside of some large ball (say of raduis R), g is zero and so $\overline{\partial} \mathcal{C}_g$ is as well. Therefore $\mathcal{C}_g$ is holomorphic on $\{z \mid ||z|| > R\}$ and $\mathcal{C}_g \equiv 0$ on $\{z \mid ||(z_2, \ldots, z_n)|| > R\}$ (by using the formula above for $\mathcal{C}_g(z)$). By the uniqueness continuation theorem, $\mathcal{C}_g \equiv 0$ on $\{z \mid ||z|| > R\}$. □

Exercise 0.51. Given $g = \sum g_j(x) dx_j$ on $\mathbb{R}^n$, where $n > 1$, such that $dg = 0$ (i.e. $\frac{\partial g_j}{\partial x_k} = \frac{\partial g_k}{\partial x_j}$) and g has compact support, then solve $df = g$ where f has compact support. Note: the assumption that $n > 1$ is necessary for topological reasons that should become apparent.

Corollary 0.52 (Hartog's: 1906)**.** *If $\Omega \subset \mathbb{C}^n$ open and $n \geq 2$, $k \subset \Omega$ compact, $\Omega \setminus k$ connected and f holomorphic on $\Omega \setminus k$, then f extends to $\tilde{f}$ holomorphic on Ω.*

Ehrenpreis: 1961. Pick $\psi \in C_0^\infty(\Omega)$ so that $\psi \equiv 1$ on a neighborhood containing k (we can do this thanks to partitions of unity - see handout). Since ψ has compact support, it must be zero near the boundary of Ω. Let

$$g = \begin{cases} (1 - \psi) f & \text{on } \Omega \setminus k \\ 0 & \text{on } k \end{cases}$$

g satisfies most of the properties we want for $\tilde{f}$, however it is not holomorphic, so we need to adjust it slightly. Let $\tilde{f} = g + u$. We need

$$0 = \overline{\partial}\tilde{f} = \overline{\partial}g + \overline{\partial}u = \overline{\partial}u + \begin{cases} -f\overline{\partial}\psi & \text{on } \Omega \setminus k \\ 0 & \text{on } k \end{cases}$$

Solving for $\overline{\partial}u$ we get:

$$\overline{\partial}u = \begin{cases} f\overline{\partial}\psi & \text{on } \Omega \setminus k \\ 0 & \text{on } k \cup \mathbb{C}^n \setminus \Omega \end{cases}$$

We can solve this such that u has compact support. By unique continuation, $u \equiv 0$ on the unbounded component of $\mathbb{C}^n \setminus supp\overline{\partial}\psi$, which includes a non-empty subset of $\Omega \setminus k$. So $\tilde{f} = f$ on a non-empty subset of $\Omega \setminus k$. Since $\Omega \setminus k$ is connected, $\tilde{f} = f$ on $\Omega \setminus k$ □

When $n > 1$:
So if f is holomorphic on $r_1 < ||z|| < r_2$, then f extends to a holomorphic function on $||z|| < r_2$.
A real analogue is: If f is locally constant on $r_1 < ||z|| < r_2$, then f extends to a constant function on $||z|| < r_2$.
Both of these fail when $n = 1$.

Definition 0.53. A map f is **biholomorphic** if f is holomorphic and it has a holomorphic inverse.

Suppose we have a biholomorphic map between a bi-disk with a Hartog's figure and a bi-disk in Ω with a Hartog's figure, where $n > 1$. The biholomorphic map distorts the Hartog's figure.
This was proven successful by Merker and Porten in 2007 (and proved unsuccessfully many times).

Lecture 9. January

Corollary 0.54. *Let $\Omega \subset \mathbb{C}^n$ ($n \geq 2$) be a connected open set and f holomorphic on Ω. Then:*

(1) $\Omega \setminus f^{-1}(0)$ *is connected*
(2) $f^{-1}(0)$ *is non-compact or empty.*

Proof. (1) (outline) Given $z, w \in \Omega \setminus f^{-1}(0)$ we can join these points by polygonal in Ω. Since $f^{-1}(0)$ has no interior, we can preturb the vertices so that all vertices lie outside of $f^{-1}(0)$. Each segment ha a parameter $[0,1] \longrightarrow \Omega$ given by $t \mapsto z_j + t(z_{j+1} - z_j)$. Complexify t, get $f(z_j + t(z - j + 1 - z_j)$ with isolated zeros. This comes from the one variable result where the zeroset is isolated.

(2) Let $h = \frac{1}{f}$ on $\Omega \setminus f^{-1}(0)$ connected. If $f^{-1}(0)$ is compact, then h extends to $\tilde{h}$ holomorphic on Ω by the Hartog's Extension Theorem. Then $\tilde{h}f = 1$ on $\Omega \setminus f^{-1}(0)$.so that $\tilde{h}f = 1$ on Ω, so $f^{-1}(0) =$.

□

Corollary 0.55. *If $\Omega \subset \mathbb{C}^n$ is a bounded open set, $n \geq 0$, and $f : \overline{\Omega} \longrightarrow \mathbb{C}$ is continuous on $\overline{\Omega}$ and holomorphic on Ω, then $f(\overline{\Omega}) = f(b\Omega)$.*

Remark 0.56. This is stronger than the maxmimum modulus principle from one-variable, which states that: $max_{\overline{\Omega}}|f| = max_{b\Omega}|f|$. The open mapping theorem on one variable tells us that $bf(\Omega) \subset f(b\Omega)$.

Proof. Pick $z_0 \in \Omega$. We need to find a point in $z \in b\Omega$ such that $f(z) = f(z_0)$. The set $\{z \in \Omega \mid f(z) - f(z_0) = 0\}$ is non-compact by the previous corollary (and a connectedness argument since we did not assume this set was connected) so it must hit the boundary of Ω. □

Let $\Omega \subset \mathbb{C}$ open.
$b\Omega C^k \overset{\text{def}}{\Leftrightarrow} b\Omega$ is C^k 1-dimensional submanifold of $\mathbb{C} = \mathbb{R}^2 \Leftrightarrow \forall z_0 \in b\Omega \exists C^k$ parameter $z : (t_0 - \epsilon, t_0 + \epsilon) \longrightarrow b\Omega$ such that $z(t_0) = z_0$ and $z'(t_0) \neq 0 \Leftrightarrow b\Omega$ is locally a C^k graph $(y = \psi(x)$ or $(x = \varphi(y))$.
We use the inverse function theorem to prove the right direction and set the graph as the parameter for the left direction.
It is also equivalent that $\overline{\Omega}$ is locally diffeomorphic to half-space - we can see this by sending $(x, y) \mapsto (x, y - \psi(x))$.
Also equivalent: $\exists \rho \in C^k(\mathbb{C})$ such that $\Omega = \{z | \rho(z) < 0\}$ and $d\rho \neq 0$ on $b\Omega$
(This last remark implies the third one by the Implicit function theorem).
Let $\rho = \pm$ distance to $b\Omega$ near $b\Omega$ and patch it together elsewhere.
Recall that if $g \in C^k(\overline{\Omega}0 \Rightarrow \mathcal{C}_\Omega(g) \in C^k(\Omega)$. How does $\mathcal{C}_\Omega g$ behave at boundary points?

Example 0.57. Suppose $g = 1, \Omega = \Delta$. Check that $\mathcal{C}_\Delta g = \begin{cases} \overline{z} & \text{on } \overline{\Delta} \\ \frac{1}{z} & \text{on } \mathbb{C} \setminus \Delta \end{cases}$. This is clearly continous on $\mathbb{C}$ but not C^1 on the boundary.

Theorem 0.58. *Suppose that $b\Omega$ is C^4 and that $g \in C^3(\overline{\Omega})$. Then $\mathcal{C}_\Omega g \in C^1(\overline{\Omega})$.*

Lemma 0.59. *$\exists \varphi \in C^2(\overline{\Omega})$ such that $\overline{\partial}(\rho\varphi) - g$ vanishes to first order on $b\Omega$.*

Proof. (Of Theorem Assuming lemma). Let $\tilde{g} = \begin{cases} g - \overline{\partial}(\rho\varphi) & \text{on } \overline{\Omega} \\ 0 & \text{on } \mathbb{C} \setminus \overline{\Omega} \end{cases}$. Then $\tilde{g} \in C^1(\mathbb{C})$.
$\mathcal{C}_\Omega g = \mathcal{C}_\mathbb{C}\tilde{g} + \mathcal{C}_\Omega(\overline{\partial}(\rho\varphi))$. The first term is in $C^1(\mathbb{C})$ and the second term, by Cauchy-Green's theorem, is $\rho\varphi$. Therefore $\mathcal{C}_\Omega$ is C^1. □

Lecture 10. January 30, 2009

Cauchy Problem for $\overline{\partial}$
Given: γ curve in $\mathbb{C}$ and g (a (0,1)-form $g(z)d\overline{z}$) curve near γ
To Solve: $\begin{cases} \overline{\partial} f \equiv g & \text{near } \gamma \\ f \equiv 0 & \text{on } \gamma \end{cases}$
Rewrite $\overline{\partial}(\rho\zeta) = g$, defining function for γ

This problem cannot always be solved. For instance, let $\gamma = \mathbb{R} \subset \mathbb{C}, \rho = y$ and $g = \frac{1}{x^3} e^{-\frac{1}{x^2}} d\overline{z}$ (implicitly we are defining $g(0) = 0$).
$e^{-\frac{1}{x^2}}$ is a solution to this and any two solutions differ by a holomorphic function, so:
$y\zeta(z) = e^{-\frac{1}{x^2}} + h(z)$, where $h(z)$ is holomorphic. This is holomorphic near the origin so that $h(x) = -e^{-\frac{1}{x^2}}$ but this cannot happen near 0.
The problem is always solvable when γ, g are real-analytic (see the homework), but these conditions are too strong to be desirable, so instead we will relax the problem.
Relaxed Cauchy Problem: Settle for $\overline{\partial}(\rho\zeta) - g$ vanished to some fized order on γ

Lemma 0.60. *Let $\Omega \subset \mathbb{C}$ be open and bounded, $b\Omega C^{k+3}, g \in C^{k+2}(\overline{\Omega}) \Rightarrow \exists \varphi \in C^{k+1}(\overline{\Omega})$ such that $\overline{\partial}(\rho\varphi) - g$ vanishes to k^{th} order along γ.*

Theorem 0.61. *If $\Omega.g$ as above, then $\mathcal{C}_\Omega g \in C^k(\overline{\Omega})$.*

Proof. (Of Lemma). Let's assume that $k = 1$. Suffices to get $\overline{\partial}(\rho\varphi) = g + \rho^2\eta$, where $\eta \in C^1(\overline{\Omega})$.

$$\overline{\partial}(\rho\varphi) = \varphi\overline{\partial}\rho + \rho\overline{\partial}\varphi$$

Get $\varpi\overline{\partial}\rho = g$ on γ and $\varphi = \frac{g}{\overline{\partial}\rho} + \rho\psi$
$(\frac{g}{\overline{\partial}\rho} + \rho\psi)\overline{\partial}\rho + \rho\overline{\partial}(\frac{g}{\overline{\partial}\rho} + \rho^2\overline{\partial}\psi + \rho\psi\overline{\partial}\rho = g + \rho^2\eta$
$\Rightarrow \psi = -\frac{\overline{\partial}(\frac{g}{2\overline{\partial}\rho}}{}$
$\eta = \overline{\partial}\psi$
All of these formulas work, but we have to make sure that the denominator does not vanish. Along the curve itself, $\overline{\partial}\rho$ is nonzero and so it is nonzero on a neighborhood of the curve. Since Ω is bounded we know that ρ must attain a minimum or maximum awa from the boundary, so that at some point away from the boundary we will have that $\overline{\partial}\rho = 0$. To remove this problem, using partitions of unity split g into $g_1 + g_2$ where $g_1 \equiv 0$ near the zeroes of $\overline{\partial}\rho$ and $g_2 \equiv 0$ near $b\Omega$. Apply the above to g_1 and ignore g.
Now suppose $k = 2$. We get φ_1 as in $k = 1$ and $\varphi_2 = \varphi_1 + \rho^2 q$, what must q do on $b\Omega$?
Exercise: Finish the proof for general k using induction. □

Let's move to $\mathbb{C}^n$:
Let S be a smooth real hypersurface (of real dimension $2n-1$) in $\mathbb{C}^n$.
$S = \{z \mid \rho(z) = 0\}$, where ρ is $\mathbb{R}$-valued and $\partial\rho \neq 0$ on S.
$g = \Sigma g_j(z) d\overline{z}_j$, smooth in a neighborhood of S, and $\overline{\partial}g = 0$.
Also assume that $g \wedge \overline{\partial}\rho = 0$ on S.
Cauchy Problem:
Solve $\overline{\partial}(\rho\varphi) = g$ near S. **Relaxed Cauchy Problem:**
Solve $\overline{\partial}(\rho\varphi) - g$ vanishes to first order on S.
As above, we need: $\varphi\overline{\partial}\rho = g$ on S
We need $g = \lambda\overline{\partial}\rho$ on S, where λ is a scalar function (i.e. $g \wedge \overline{\partial}\rho = 0$ on S).
Then $g = \lambda\overline{\partial}\rho + \rho\tau$, where $\rho\tau$ is an error term. Take $\varphi = \lambda + \rho v$.
Goal: $\overline{\partial}(\rho\varphi) - g = \rho^2\eta$
Plugin to get that $\rho(\overline{\partial}\lambda + 2v\overline{\partial}\rho - \tau) = \rho^2(\eta - \overline{\partial}v)$.
We can divide by ρ in the above equation so it simplifies to: $\overline{\partial}\lambda + 2v\overline{\partial}\rho - \tau = \rho(\eta - \overline{\partial}v)$.
This is hopeless unless $\overline{\partial}\lambda - \tau$ is a scalar multiple of $\overline{\partial}\rho$ along S.

$$\begin{aligned} 0 &= \overline{\partial}g \\ &= \overline{\partial}\lambda \wedge \overline{\partial}\rho + \overline{\partial}\rho \wedge \tau + \rho\overline{\partial}\tau \\ &= (\overline{\partial}\lambda - \tau) \wedge \overline{\partial}\rho + \rho\overline{\partial}\tau \\ &= \begin{cases} (\overline{\partial}\lambda - tau) \wedge \overline{\partial}\rho = 0 & \text{on } S \\ \overline{\partial}\lambda - \tau = \mu\overline{\partial}\rho + \rho\sigma \end{cases} \end{aligned}$$

Apply this to: $\overline{\partial}\lambda + 2v\overline{\partial}\rho - \tau = \rho(\eta - \overline{\partial}v)$ (which we found above), and we get that $\mu[\overline{\partial}]\rho + 2v\overline{\partial}\rho = \rho(\eta - \overline{\partial}v - \sigma)$.
Take $v = -\frac{\mu}{2}$ and $\eta = \overline{\partial}v + \sigma$.

Lecture 11. February 2, 2009

Lemma 0.62. *Give S a C^4 real hypersurface in $\mathbb{C}^n$, ρ defining function for S , g a $C^3(0,1)$-form near S, $\overline{\partial}g = 0$, and g a scalar multiple of $\overline{\partial}\rho$ on S. Then $\exists\varphi C^2$ near S such that $\overline{\partial}(\rho\varphi) - g$ vanishes to first order on S.*

Note that for us, smooth is something between C^1 and C^∞, but the precise meaning is negotiable.

Topic: Boundary Value of Holomorphic Functions
Let $\Omega \subset \mathbb{C}$ be an open, bounded set. Suppose $b\Omega$ is smooth and connected, $f : b\Omega \longrightarrow \mathbb{C}$. When does f extend to a holomorphic functiion on Ω?
Special Case: Assume that $b\Omega$ and f are real-analytic (so f is given by a real power series). Then we can extend f to be holomorphic in some neighborhood of $b\Omega$.
Let γ_1 be a curve on the inside of $b\Omega$ and γ_2 be a curve on the outside of $b\Omega$, where $\gamma_1, \gamma_2, b\Omega$ are all curves oriented in the same direction. Then

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)d\zeta}{\zeta - z} \\ &\equiv f_2(z) - f_1(z) \end{aligned}$$

So f_2 is holomorphic inside γ_2 and f_1 is holomorphic outside γ_1 and vanishes at infinity.
$f_1 \equiv 0 \Rightarrow f$ extends to a holomorphic function on Ω since γ_2 contains Ω. The other implication ($\Leftarrow$) follows by Cauchy's Theorem.
$f_1 \equiv 0 \Leftrightarrow \int_{b\Omega} \frac{f(\zeta)d\zeta}{\zeta - z} = 0 \forall z \in \mathbb{C} \setminus \overline{\Omega} \Leftrightarrow 0 = \int_{b\Omega} f(\zeta)\zeta^n d\zeta \forall n \geq 0$

Exercise 0.63. $\sum_{n\geq 0} \frac{(-1)^n}{2\pi i} z^{-n} \int_{b\Omega} f(\zeta)\zeta^n d\zeta = \int_{b\Omega} \frac{f(\zeta)d\zeta}{\zeta - z} = 0 \forall z \in \mathbb{C} \setminus \overline{\Omega}$. *Note that this is called a moment.*

Theorem 0.64. *Let $n \geq 2$. Suppose that $\Omega \subset \mathbb{C}^n$ is bounded and open, $b\Omega$ is connected and C^4, $f \in C^4(\overline{\Omega})$,*

$$\dagger \quad \overline{\partial} f \text{ is a scalar multiple of } \overline{\partial}\rho \text{ on } b\Omega,$$

Then $\exists h \in C^1(\overline{\Omega})$ such that h is holomorphic on Ω and $h = f$ on b
Omega.

Proof. (Reference: see Range 2002 Math Intelligencier for the history on this theorem)
Use the lemma stated at te begining of this lecture. Let $\overline{\partial}(\rho\varphi) = \overline{\partial} f + \rho^2 \eta$. Let

$$\beta = \begin{cases} \rho^2\eta = \overline{\partial}(\rho\varphi - f) & \text{on } \overline{\Omega} \\ 0 & \text{on } \mathbb{C}^n \setminus \overline{\Omega} \end{cases}$$

Then β is C^1 on $\mathbb{C}^n$ and β has compact support. Also notice that $\overline{\beta} \equiv 0$.
Let $\alpha \in \mathcal{C}_{\mathbb{C}^n}\beta$ (as defined 3 lectures ago). Then α is C^1 and $\overline{\partial}\alpha = \beta$.
We need $n \geq 2$ since in this case α has compact support. Then $\alpha \equiv 0$ on $\mathbb{C}^n \setminus \overline{\Omega}$. Note that we are combining connectedness, compact support, and the unique continuation principle.
Then $\alpha \equiv 0$ on $b\Omega$. Let $h = f + \alpha - \rho\varphi$ on $\overline{\Omega}$. Then $\overline{\partial} h \equiv 0$ on $\Omega \Rightarrow h$ is holomorphicon Ω.
$h \equiv f$ on $b\Omega$. ☐

Common Trick/Method:
Get holomorphic functions as the difference of two different solutions of a $\overline{\partial}$-problem.
The assumption † is necessary: $h = f + \rho g$. Apply $\overline{\partial}$:
$0 = \overline{\partial} f + \rho\overline{\partial} g + g\overline{\partial}\rho$. On $b\Omega$: $\overline{\partial} f = -g\overline{\partial}\rho$

Example 0.65. Let $\Omega = B^2$ (the unit ball in $\mathbb{C}^2$) and $\rho = z_1\overline{z_1} + z_2\overline{z_2} - 1$.
We need $\overline{\partial} f = f_{\overline{z_1}} d\overline{z_1} + f_{\overline{z_2}} d\overline{z_2}$ to be a scalar multple of $\rho_{\overline{z_1}} d\overline{z_1} + \rho_{\overline{z_2}} d\overline{z_2}$ on bB^2.
$\rho_{\overline{z_1}} d\overline{z_1} + \rho_{\overline{z_2}} d\overline{z_2} = z_1 d\overline{z_1} + z_2 d\overline{z_2}$
So we need their wedge product to be zero: $z_2 f_{\overline{z_1}} - z_1 f_{\overline{z_2}} = 0$ on $b\Omega$.
Now let $\Omega = B^n$ (the unit ball in $\mathbb{C}^n$). Then $\rho = z_1\overline{z_1} + \cdots + z_n\overline{z_n} - 1$.
We need $z_j f_{\overline{z_k}} - z_k f_{\overline{z_j}} = 0$ on bB^n for $j < k$.
If we are not concerned about the denominators vanishing we can simplify this as:

$$\frac{f_{\overline{z_1}}}{z_1} = \cdots = \frac{f_{\overline{z_n}}}{z_n}$$

Near any particular point this can be describe with $n - 1$ equations.

Lecture 12. February 4, 2009

Let $b\Omega\mathbb{C}^n$ be bounded, $b\Omega$ be smooth and connected, then te boundary values of holomorphic functions is characterized by PDEs for $n \geq 2$.
Another Approach: Suppose that f is on $b\Omega$ extends to a holomorphic function h on Ω, which is smooth on $\overline{\Omega}$.

$$df_p : T_p b\Omega \longrightarrow \mathbb{C} \text{ extends to } dh_p : T_p\mathbb{C}^n \longrightarrow \mathbb{C}$$

‡ dh_p is $\mathbb{C}$-linear so df_p must be $\mathbb{C}$-linear on the maximal $\mathbb{C}$-subspace of $T_p b\Omega$ (i.e. on $H_p(b\Omega) \equiv T_p b\Omega \cap JT_p b\Omega$).
So we need that $df(JX) = idf(X) \forall X \in H_p(b\Omega)$.
Let $X = (a, b, \ldots) \in \mathbb{R}^{2n} = \mathbb{C}^n$ and identify it with $a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial y_1} + \ldots$.
Then $Xf = df(X) =$ derivative of f in the direction of X.
So $(JX)f = df(JX) = idf(X) = iXf \forall X \in H_p(b\Omega) \Rightarrow (X + iJX)f = 0 \forall X \in H_p(b\Omega) \Rightarrow X_p = 0 = (JX)_p \forall p \in b\Omega$.

Exercise 0.66. $\{X + iJX\} = \{\sum c_j \frac{\partial}{\partial \overline{z_j}}$

$$\begin{aligned} f \text{ satisfies: } \ddagger &\Leftrightarrow \sum c_j \frac{\partial f}{\partial \overline{z_j}} = 0 \text{ when } \sum c_j \frac{\partial \rho}{\partial \overline{z_j}} = 0 \\ &\Leftrightarrow \overline{\partial} f \text{ is a scalar multiple of } \overline{\partial}\rho \\ &\Leftrightarrow^{\text{def}} f \text{ satisfies the tangential CR-equations} \\ &\Leftrightarrow^{\text{exercise}} df \wedge dz_1 \wedge \ldots \wedge dz_n|_{b\Omega} = 0 \end{aligned}$$

Theorem 0.67. *$\Omega \subseteq \mathbb{C}^n$ bounded, $b\Omega$ smooth and connected with $n \geq 2$, then f on $b\Omega$ extends to a holomorphic function on $\Omega \Leftrightarrow f$ satisfies the tangential CR-equation.*

More generally: If M is a real submanifold of $\mathbb{C}^n$ and $p \in M$, then $H_pM = T_pM \cap JT_pM$ is a maximal $\mathbb{C}$-subspace of T_pM.

Definition 0.68. M is a **CR-manifold** if $\dim H_pM$ is constant.

Definition 0.69. M is **totally real** if $\dim H_pM = 0$.

Definition 0.70. M is a **generic** CR-manifold if
Def. 1 $\dim H_pM = max\{\dim_{\mathbb{R}} M - n, 0\}$
Def. 2 $\dim H_pM = \dim_{\mathbb{R}} M - n$

Definition 0.71. Let M be a CR-manifold then $f : M \longrightarrow \mathbb{C}$ (atleast C^1) is **CR** if d_pf is $\mathbb{C}$-linear on each H_pM.

Equivalently, $(X + iJX)f = 0 \forall X \in H_pM$.

Theorem 0.72 (Tool Baouendi-Treves Approximation Theorem)**.** *Given $p \in U \subset M \subset \mathbb{C}^n$, where U is open and M is a Def. 1 generic CR-manifold, and $f : U \longrightarrow \mathbb{C}$ is CR. Then $\exists\{f_j\}$ entire functions such that $f_j \longrightarrow f$ uniformly on V where $p \in V \subseteq U$ open.*

Proof. Tools to use are: Gaussian convolution, Stoke's, Cauchy theory for $\overline{\partial}$. A reference is: Boggess "CR Manifolds and ...". We will omit the proof here, but it would take about one full lecture. □

Example 0.73 (Special Case). Suppose that $M = bB^2$ and $p = (1, 0)$.Choose an open set V that contains a neighborhood of p. In particular, we can choose a neighborhood of the form $bB^2 \cap \{Rez_2 > 1 - \epsilon\} \subset M$. Fix z_2 with $Re(z_2) > 1 - \epsilon$. Suppose we have a sequence of functions $\{f_j(z_1, z_2\}$ that converge uniformly to $f(z_1, z_2)$ for $|z_1| = \sqrt{1 - |z_2|^2}$

and to $h(z_1, z_2)$ for $|z_1| \leq \sqrt{1-|z_2|^2}$. Use the uniform Cauchy-criterion and the maximum principle. Then $f_j \longrightarrow h$ uniformly on $\overline{B^2} \cap \{Re(z_2) > 1-\epsilon\}$ and h is holomorphic on $B^2 \cap \{Re(z_2) > 1-\epsilon\}$ such that $h|_{bB^2} = f$. So a CR function is any neighborhood of $p \in bB^2$ has local 1-sided holomorphic extension.
This method of proof is called the "disk method."

Exercise 0.74. Let $M = \{(z_1, z_2, z_3) \mid Re(z_3) = |z_1|^2 - |Z_2|^2\}$. Show that any CR function defined near $0 \in M$ has a holomorphic extension to a full neighborhood of 0.

Lecture 13. February 6, 2009

Recall: Given $T : \mathbb{C}^n \longrightarrow \mathbb{C}^k$ is $\mathbb{R}$-linear, then T is $\mathbb{C}$-linear $\Leftrightarrow$ the graph of T is a $\mathbb{C}$-subspace of $\mathbb{C}^{n+k}$.

Corollary 0.75. *Given a C^1 function $f : \Omega \longrightarrow \mathbb{C}^k$, where $\Omega \subset \mathbb{C}^n$ is open, then:*

$$f \text{ is holomorphic} \iff \text{each } T_p(\text{graph of } f) \text{ is a } \mathbb{C}\text{-subspace of } \mathbb{C}^{n+k}$$

So the graph of the derivative map gives the tangent space to the graph of the original map. Now consider a submanifold $M \subset \mathbb{C}^2$ such that $\dim_{\mathbb{R}} M = 2$ and T_pM is complex $\forall p \in M$. Pick a point $p \in M$ and assume that $T_pM \neq \{0\} \times \mathbb{C}$. Then by the implicit fuction theorem, M is a graph $\{(z, w) \mid w = f(z)\}$ near p. Therefore by the corollary, f is holomorphic.

Exercise 0.76. Show that M satisfies the holomorphic version of all conditions for 1-manifolds from the lecture on January 28. Need to show that there exists a global defining function, but omit this for now since it is a more advanced result.

Similar results hold whenever M is a complex submanifold of $\mathbb{C}^n$ (i.e. all T_pM are complex).
Laurent Series:
Let $A_j = \{z \in \mathbb{C} \mid r_j < |z| < R_j\}$ and f be holomorphic on a neighborhood of $\overline{A_1} \times \cdots \times \overline{A_n}$. Then:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{bA_1 \times \cdots \times bA_n} \frac{f(\zeta) d d\zeta_1 \wedge \cdots \wedge d\zeta_n}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)}$$

Use the same method as for the polydisk case. Note that $bA_1 \times \cdots \times bA_n$ has 2^n connected components since each A_i has two connected boundary components. Each component is the product of n circles, so it is an n-dimensional torus.
It is an exercise to show that the above $f(z)$ becomes:

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha, \text{ this is almost uniform on } A_1 \times \ldots A_k$$

We get this result by using the following, where $z_j \in A_j$:

$$\frac{1}{\zeta_j - z_j} = \begin{cases} -\frac{1}{z_j} \frac{1}{1 - \frac{\zeta_j}{z_j}} & \text{on the inner boundary} \\ \frac{1}{\zeta_j} \frac{1}{1 - \frac{z_j}{\zeta_j}} & \text{on the outer boundary} \end{cases}$$

So the formula for determining c_α is:

$$c_\alpha = \frac{1}{(2\pi i)^n} \int_{|\zeta_j| = \gamma_j} \frac{f(\zeta) d\zeta_1 \wedge \cdots \wedge d\zeta_n}{\zeta^{\alpha+1}}$$

The γ_j must stay between r_j and R_j, but we can slide the γ_j around.
Proof 1: Quote the 1-dimensional result and use it repeatedly for each of the j.
Proof 2: Suppose we have two different γ_j's (i.e. γ_j and $\tilde{\gamma_j}$). Then a curve between these two points on the z_1, z_n plane is an $n+1$-dimensional manifold bounded by two tori, so we can apply Stoke's theorem. So we can consider $\partial(\frac{f(\zeta) d\zeta_1 \wedge \cdots \wedge d\zeta_n}{\zeta^{\alpha+1}}) = 0$. So the integral does not depend on the specific γ_j.

Definition 0.77. A **Reinhardt domain** is a connected open subset of $\mathbb{C}^n$ that is invariant under the map:

$$(z_1, \ldots, z_n) \mapsto (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \equiv e^{i\theta} z$$

Theorem 0.78. *If f is holomorphic on a Reinhardt domain Ω, then $f(z) = \sum c_\alpha z^\alpha$ is almost uniform on Ω, where c_α is given as above.*

Proof. **1** Quote the 1-dimensional result
2 Repeat the Stoke's theorem argument
Almost uniform convergence: We can cover any compact subset K with a finite subcover and then use earlier results for convergence. □

Suppose that Ω intersects an axis so that $\Omega \cap \{z_j = 0\} \neq \emptyset \Rightarrow c_\alpha = 0$ when $\alpha_j < 0$.
Proof 1: Use Cauchy on a disk times a polyannulus.
Proof 2: Use Stoke's theorem again to $c_\alpha = 0$ when $\alpha_j < 0$. On a coordinate axis we would be dividing by $\zeta_i^{\alpha_1+1}$ so that if $\alpha < 0$ then we are not dividing by zero but instead multiplying by it.
Re-explaining:
If we are considering $n = 2$, then a point on the z_i axis corresponds to a circle, the origin corresponds to a point, and a point not on an axis corresponds to a circle times a circle (i.e. a torus). What we are using to show that if $\alpha_1 < 0$, then $c_\alpha < 0$ is two things. One is that since $\alpha < 0$, there will be no problems dividing by ζ^{α_1+1}, where $\alpha_1 = 0$, since we will instead by multiply be zero. The other thing is that our function is holomorphic, so using Stoke's theorem the integral will be zero.

Lecture 14. February 9, 2009

$$\begin{aligned}\{f_j\} \text{ converges uniformly on } E &\Leftrightarrow \forall \epsilon > 0 \exists N \text{ such that } |f_j(x) - f_k(x)| < \epsilon, x \in E, j, k > N \\ &\Leftrightarrow \{f_j\} \text{ uniformly Cauchy on } E\end{aligned}$$

$$\begin{aligned}\sum f - j \textbf{ uniformly convergent on } E &\Leftrightarrow \forall \epsilon > 0 \exists N \text{ such that } |f_j(x) - f_k(x)| < \epsilon, x \in E, N < j < k \\ &\Leftrightarrow \sum f_j \text{ uniformly Cauchy on } E\end{aligned}$$

Rienhardt domain is a connected open set in $\mathbb{C}^n$ that is invariant under all coordinate rotations.
Flip it: start with $\sum c_\alpha z^\alpha$ - where does this converge?
Let $A = \{z \in \mathbb{C}^n \mid \sum |c_\alpha z^\alpha| < \infty\}$ - A is rotation invariant.
Let $\Omega = Int(A) =$ interior of A - A is rotation invariant
Let $L(A) = \{(\log|z_1|, \ldots, \log|z_n|) \in \mathbb{R}^n \mid z \in A, z_i \neq 0\} = \{x \in \mathbb{R}^n | \sum |c_\alpha| e^{\alpha x} \equiv \psi(x) < \infty\}$. Then $\psi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\infty\}$. ψ is convex (i.e. $\psi(tx_1 + (1-t)x_2) \leq t\psi(x_1) + (1-t)\psi(x_2) \forall x_i \in \mathbb{R}^n, 0 \leq t \leq 1$). Convexity of ψ follows from the convexity of the exponential function in one variable. Therefore $L(A)$ is convex (i.e. for any two points in $L(A)$, the line joining them must also be in $L(A)$) and so $L(\Omega) = L(Int(A)) = Int(L(A))$ is also convex. Therefore Ω is "logarithmically convex." It follows that Ω is a Reinhardt domain.
If $c_\alpha \neq 0$ for some α with $\alpha_j < 0$, then $\Omega \cap \{z_j = 0\} = \emptyset$. Otherwise, if for some j we have $c_\alpha < 0$ when $\alpha_j < 0$, then ψ increasing function of x_j

Therefore the projection of Ω, $\text{Proj}_{\{z_j=0\}} \Omega \subset \Omega$.
We will show that the properties:

- Let $\Omega = Int(A) =$ interior of A - A is rotation invariant
- Ω is "logaritimically convex"

- $\text{Proj}_{\{z_j=0\}}\Omega \subset \Omega$

Characterize "domains of convergence"

We want $f(z) = \sum c_\alpha z^\alpha$ holomorphic on Ω. It suffices to show that $\sum c_\alpha z^\alpha$ is uniform in a neighborhood of each $z_0 \in \Omega$.
Pick $x_0 \in L(\Omega)$. Since $L(\Omega)$ is convex, we can get a cube surrounding x_0 in $L(\Omega)$. $\sum |c_\alpha| e^{\alpha x}$ converges uniformly on vertices. Since $L(\Omega)$ is convex, is the unifom Cauchy critierion is satisfied on the vertices, then it must also be satisfied on the edges and by the same reasoning it is satisfied on the faces (2-dimensional). Continuing by induction, the sum must converge uniforrmly on the verticles$\Rightarrow$ edges$\Rightarrow$ faces$\Rightarrow \cdots \Rightarrow$ cube. Therefore $\sum c_\alpha z^\alpha$ converges absolutely uniformly on the poly-annulus containing z_0.

Remark 0.79. Absolutely convergent sums, converge to the same thing even if the terms are rearranged, while conditionally convergent sums can converge to different things when the terms are rearranged.

So far we have aassume absolute convergence, but what about conditional convergence?
Let $B = \{z \mid \sup\{|c_\alpha z^\alpha|\} < \infty\}$.

Exercise 0.80. Prove Abel's lemma: that the interior of B is contained in A (so $Int(B) = \Omega$).

Example 0.81. Let $|zw| < 1$. We draw a picture of this restricted to $\mathbb{R}^2$ and by looking at the logarithms so that $\log z < -\log w$.
$\sum_{i=0}^{\infty} z^j w^j = \frac{1}{1-zw}$ converges on Ω but nowhere else.

Example 0.82. Suppose $|z|^2|w|^{1+\sqrt{5}} < 1$. Then we can draw the same types of pictures as above. Our first attempt to find a series that converges only on Ω, might be to consider $\sum_{j=0}^{\infty} z^{2j} w^{(1+\sqrt{5})j}$, but this does not work since we get non-integer exponents. A second attempt might be:
Due to the density of the rationals in $\mathbb{R}$, the value of $j(1+\sqrt{5})$, where $j \in \mathbb{N}$, will get close to be an integer infinitely many times, so we will through out the terms where this is not close to an integer. A power series that does converge precisely on Ω, $zw + z^2w^3 + z^5w^8 + z^13w621 + \ldots$ (note that the exponents are coming from the Fibonacci series).

Lecture 15. February 11, 2009

Theorem 0.83. *Given an open set $\Omega \subset \mathbb{C}^n$, then Ω is the domain of convergence for some Laurent series if and only if the following hold:*

1. *Ω is Reinhardt*
2. *Ω is logarithmically convex*
3. *$\forall j, \Omega \cap \{z_j = 0\} = \emptyset$ or $Proj_{\{z_j=0\}}\Omega \subset \Omega$.*

Proof. ($\Rightarrow$) Proved last lecture
($\Leftarrow$) Will prove later ◻

Exercise 0.84. *(Weak form of ($\Leftarrow$) in above theorem)* Given $\Omega_1 \subsetneq \Omega_2$ Reinhardt, where Ω_1 satisfies conditions 1 and 2, then $\exists \sum c_\alpha z^\alpha$ converges on Ω_1 by not on Ω_2.

Recall: Hartogs Figure
If we graph this on a logarithmic graph, we can see that it is not logarithmically convex, so if there is a Laurent series that converges on the Hartog's figure, it must converge on a larger domain by the previous theorem. This recovers the "Hartog's configuration theorem."

Given f holomorphic on Ω Reinhardt (and assuming that 2 and/or 3 fails), let $f = \sum c_\alpha z^\alpha$. Then $\sum c_\alpha z^\alpha$ converges on a Reinhardt domain $\tilde{\Omega} \supsetneq \Omega$, where $\tilde{\Omega}$ satisfies conditions 2 and

3. Fact: there is a minimial $\tilde{\Omega}$ satisfying the above. All holomorphic functions on Ω extend to $\tilde{\Omega}$.

Example 0.85. We now consider a Hartog's figure H with a fattened real curve (i.e. tail) extending from the figure. Every real function on $\Omega = H\cup$ tail, extends to $\tilde{\Omega} = \Delta^2\cup$ tail. Note: we know that it extends from H to Δ^2 by the Hartog's configuration theorem. However, this extension may force the function on $\tilde{\Omega}$ to be a multi-valued function since the bi-disk Δ^2 will overlap part of the tail that was disjoint from H. To fix this, we will consider Riemann surfaces.

Definition 0.86. A **Riemann domain** is a Hausdorff topological space X equipped with a local homeomorphism $\varphi : X \longrightarrow \mathbb{C}^n$.

One can think of this as a Riemann surface projecting down to $\mathbb{C}^n$ - so there can be infinitely many sheets that project down to $\mathbb{C}^n$. Every Riemann manifold is a complex manifold with extra structure.

Definition 0.87. Let X be a Riemann domain. Then a function $f : X \longrightarrow \mathbb{C}$ is holomorphic if $f \circ \varphi^{-1}$ is holomorphic for all continuous branches of φ^{-1}.

Example 0.88. Given two annuli in the $\log|z|, \log|w|$ plane that have one cut (a straight cut from the outside to the inside circle). Let X be the space where the two annuli are pasted together along both sides of the cuts in a way that produces a 2-sheeted glued object. Suppose that $f : X \longrightarrow \mathbb{C}$ is holomorphic. Then $f(z) = \sum c_\alpha z^{\alpha_1} w^{\alpha_2}$, where the c_α are local constants and, hence, constant. This is single-valued so that f does not separate points. Every holomorphic function f on X extends to a single valued function.

Lecture 16. February 13, 2009

Facts:

1. $\hat{K}_{\mathbb{R}-lin}$ convex
2. $K = \hat{K}_{\mathbb{R}-lin} \Leftrightarrow K$ convex (version of Hahn Banach Thmeorem)
3. $\hat{K}_{\mathbb{R}-lin}$ is the smallest convex set containing K
4. $\widehat{\{x_1, \dots, x_n\}}_{\mathbb{R}-lin} = \{\sum c_j x_j \mid c_j \geq 0, \sum c_j = 1\}$
5. K general $\Rightarrow \hat{K}_{\mathbb{R}-lin} = \cup_{\{x_1,\dots,x_{n+1}\}} \widehat{\{x_1, \dots, x_{n+1}\}}_{\mathbb{R}-lin}$ (carathedory)
6. Ω is Reinhardt $\Rightarrow L(\tilde{\Omega}) = \widehat{L(\Omega)}_{\mathbb{R}-lin}$

Reference: Hormader Notions of Convexity
Given $\mathcal{F}$ a family of $\mathbb{R}$-valued (or $\mathbb{C}$-valued) functions on E and $K \subset E$ then

$$\hat{K}_{\mathcal{F}} \equiv \text{ complement of } \{x \in E \mid \exists f \in \mathcal{F} \text{ such that } f(x) > f(y)\big(\text{or } |f(x)| > |f(y)|\big) \forall y \in K\}$$

If $\max_K f$ or $\max_K |f|$ exists $\forall f \in \mathcal{F}$, then we can rewrite it as:

$$\hat{K}_{\mathcal{F}} = \{x \in E \mid f(x) \leq \max_K f \big(\text{or } \hat{K}_{\mathcal{F}} = \{x \in E \mid |f(x)| \leq \max_K |f|\big) \ \forall f \in \mathcal{F}\}$$

(Note: the text inside parenthesis refers to $\mathbb{C}$-valued functions).

Proposition 0.89. *$\psi : \overline{\Delta} \longrightarrow \Omega$ continuous, holomorphic on Δ and $\psi(b\Delta) \subset K \Rightarrow \psi(\Delta) \subset \hat{K}_{Holo(\Omega)}$.*

Proof. $|f \circ \psi(z)| \leq^{maxprincipal} \max_{b\Delta} |f \circ \psi| \leq \max_K |f|$ ☐

Example 0.90. $\Omega = \Delta \times \Delta$, $K = \{(z, w_0) \mid |z| = r\} \subset \Omega$, $w_0 \in \Delta, 0 < r < 1$. Then $\hat{K}_{Holo(\Omega)} = \{(z, w_0) \mid |z| \leq r\}$ (use $z - z_0, w$).

Note that $\hat{K}_{Holo(\Omega)}$ is always closed in Ω when K is compact.

Lemma 0.91. $\hat{K}_{Holo(\Omega)} \subset \hat{K}_{\mathbb{R}-lin}$

Proof. $z_0 \notin \hat{K}_{\mathbb{R}-lin} \Rightarrow \exists l : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $l(z_0) > l(z) \forall z \in K$. Let $\tilde{l}(z) = l(z) - il(iz)$, so that this is $\mathbb{C}$-linear.

$$|e^{\tilde{l}(z_0)}| = e^{l(z_0)} > e^{l(z)} = |e^{\tilde{l}(z)}| \forall z \in K \Rightarrow z_0 \notin \hat{K}_{Holo(\Omega)}$$

□

Corollary 0.92. *If K is bounded, then $\hat{K}_{Holo(\Omega)}$ is bounded.*

We want to determine when $\hat{K}$ is guaranteed to be closed.
Let $||z||_{L^\infty} = \max |z_j|$ be the L^∞-norm.
Let $K \subset^{\text{compact}} \Omega \subset \mathbb{C}^n$, where Ω is open. $f \in Holo(\Omega)$.
Pick $0 < r < dist_{L^\infty}(K, b\Omega)$. Let $M = \{\max\{|f(z)| \mid dist_{L^\infty}(z,k) \leq r\}$ (note that this depends on r). The set of z satsifying the inequality $dist_{L^\infty}(z,k) \leq r$ is a compact set, so the maximum exists.
Cauchy Estimates: $\frac{\partial^\alpha f}{\partial z^\alpha}| \leq M\alpha! r^{-|\alpha|}$ on K.
The same estimates hold on $\hat{K}_{Holo(\Omega)}$. Take a point $z_0 \in \hat{K}_{Holo(\Omega)} \Rightarrow \sum |$ terms of the Taylor series for f at z_0. Then

$$\frac{\partial^\alpha f}{\partial z^\alpha}| \leq \sum M r^{-|\alpha|} |z - z_0|^\alpha = \frac{M}{(1 - \frac{(z-z_0)_1}{r}) \dots (1 - \frac{(z-z_0)_n}{r})}$$

$\Rightarrow$ Taylor series for f at z_0 converges on $\{z \mid ||z - z_0||_{L^\infty} < r\}$.
This is only interesting if $\{z \mid ||z - z_0||_{L^\infty} < r\} \nsubseteq \Omega$, since we know that this works already on polydisks.

Lecture 17. February 16,2009

Let $\Omega \subset \mathbb{C}^n$ be open and $K \subset \Omega$ be compact.

$$\hat{K}_{Holo(\Omega)} = \{z \in \Omega | \ |f(z)| \leq \max_K |f| \forall f \text{ holomorphic on } \Omega\}$$

Special case: $\Omega = \mathbb{C}$
Let $\tilde{K} = K \cup \{$ bounded components of $\mathbb{C} \backslash K\}$. The maximum principle implies that $\tilde{K} \subset \hat{K}$ (since the bounded components achieve their maximum on the boundary$\Rightarrow$ on K. Therefore $\tilde{K} = \hat{K}$

Proof. Let $f(z) = \frac{1}{z - z_1}$ satisfies $|(f(z_0)| > \max_K |f|$ if z_1 is close to z_0. We can move the pole:

$$\begin{aligned} \frac{1}{z - z_1} &\approx \sum_{j=0}^{M} \frac{(z_1 - z_2)^j}{(z - z_2)^j} \\ &\approx \dots \text{(repeat process)} \\ &\approx \sum_{j=0}^{M'} \frac{\alpha_j}{(z - z_n)^j} \\ &\approx \sum_{j=0}^{M''} \beta_j z_j \text{ by Taylor} \end{aligned}$$

□

Suppose that the distance $dist_{l^\infty}(\hat{K}_{Holo(\Omega)}, b\Omega) < dist_{L^\infty}(K, b\Omega)$.
Then $\exists z_0 \in \hat{K}_{Holo(\Omega)}, 0 < r < dist_{L^\infty}(K, b\Omega)$ such that $P = \{z \mid ||z - z_0||_{L^\infty} < r\} \nsubseteq \Omega$.
Every holomorphic function f on Ω extends. Let

$$X = \frac{\Omega \times \{1\} \cup P \times \{2\}}{(z,1) \sim (z,2) \Leftrightarrow z \in z_0 \text{component of } \Omega \cap P}$$

Exercise: X is Hausdorff.

Exercise 0.93. Show the same result for Euclidean distance norm instead of the L^∞ norm.

Definition 0.94. U is a unitary transformation of $\mathbb{C}^n$ if it is a $\mathbb{C}$-linear map from $\mathbb{C}^n$ to $\mathbb{C}^n$ that preserves distances)

Theorem 0.95. *Let $\Omega \subset \mathbb{C}^n$ be a (connected, - not a necessary assumption) open set and $n \geq 1$. The following are equivalent:*

1. THERE IS NO AUTOMATIC EXTENSION: *There is no Riemann domain $X \supsetneq \Omega$ such that every holomorphic function on Ω extends to X.*
2. $dist_{L^\infty}(\hat{K}_{Holo(\Omega)}, b\Omega) = dist_{L^\infty}(K, b\Omega) \forall K \subset \Omega$ *compact*
3. HOLOMORPHIC CONVEXITY: $K \subset \Omega$ *compact* $\Rightarrow \hat{K}_{Holo(\Omega)} \subset \Omega$ *compact (its compactness in $\mathbb{C}^n$ depends on whether or not $\hat{K}_{Holo(\Omega)}$ hits the boundary).*
4. DISCRETE INTERPOLATION: *Let $S \subset \Omega$ discrete, $f : S \longrightarrow \mathbb{C}$. then $\exists h$ holomorphic on Ω such that $h|_S = f$*
5. DOMAIN OF EXISTENCE: *There is a holomorphic function on Ω that cannot be extended to any larger Riemman domain $X \supsetneq \Omega$*

What does a larger Riemann domain $X \supsetneq \Omega$ mean?
There is no Riemann domain (X, φ) and $\rho : \Omega \longrightarrow X$ continuous such that $\varphi \circ \rho = Id$ and $\rho(\Omega) \neq X$.

Exercise 0.96. Suppose that $\Omega \subset \mathbb{R}^n$ open. Then Ω is convex $\Leftrightarrow \hat{K}_{\mathbb{R}-linear}$ is a compact subset of Ω when $K \subset \Omega$ is compact.

Proof. (5) $\Rightarrow$ (1) is clear
(1) $\Rightarrow$ (2) we discussed earlier this lecture that not (2) $\Rightarrow$ not (1).
(2) $\Rightarrow$ (3) Let $K \subset \Omega$ compact $\Rightarrow \hat{H}_{Holo(\Omega)}$ bounded and closed in Ω with no limit points on $b\Omega$, hence closed in $\mathbb{C}^n$
(3) $\Rightarrow$ (4) Let $S = \{z_k\}_{k=1}^\infty \subset \Omega$ be discrete and let $w_k = f(z_k)$. Let $k_j = \{z \in \Omega \mid ||z|| \leq j, dist(z, b\Omega) \geq \frac{1}{j}\}$. Then each k_j is compact, $k_1 \subset k_2 \subset k_3 \subset \ldots$, and every compact subset of Ω is contained in some k_j so that $\hat{k_1} \subset \hat{k_2} \subset \ldots$ are all compact. Let $h = \sum g_k$, where g_k are holomorphic on Ω and $g_k(z_k) = w_k - \sum_{l=1}^{k-1} g_l(z)$, so that $g_j(z_k) = 0 \forall j > k$. We can construct such functions as these, but we need to make sure that they converge. $|g_k| \leq 2^{-k}$ on K. We need $z_k \notin \hat{K}_{\alpha(K)}, \alpha(K) \to \infty$. Let $\alpha(K) = \max\{m \mid z_j \notin \hat{K}_m \forall j \geq m\}$. Assume that we have already constructed $g_1, \ldots, g_{k-1}$. To construct g_k, we need to find p_k holomorphic on Ω such that

$$p_k(z_j) = \begin{cases} w_k - \sum_{l=1}^{k-1} g_l(z_k) & j = k \\ 0 & j < k \end{cases}$$

and U_k is holomorphic on Ω such that $U_k(z_k) = 1, \max |U_k| < 1, K_{\alpha(k)}$. Take $g_k = p_k U_k^{M_k}$.
(4) $\Rightarrow$ (5) **General Plan** Choose $S = \{z_j\} \subset \Omega$ discrete clustering out all boudary points. Solve $h(z_j) = j, h$ holomorphic on Ω. h is unbounded near boundary points. This is enough to show that you cannot extend to any larger open set, but this does not rule out extending to a larger Riemann domain. In order to rule out the latter possibility, pick $\{\zeta_k\}$ a countable dense subset of $b\Omega$. Consider components of $V_n = \Omega \cap B(\zeta_k, \frac{1}{n}), k, n \in \mathbb{N}$. Let $V_1, V_2, \ldots$ with each appearing infinitely often. Pick $z_j \in V_j$ such that $dist(z_j, b\Omega) = \frac{1}{j}$ and $z_{j_1} = z_{j_2} \Leftrightarrow j_1 = j_2$. $\{z_j\}$ is a discrete set in Ω. Each V_l contains infinitely many z_j. Solve

$h(z_j) = j$. Claim: h does not extend.

$$\begin{array}{ccc} & & X \\ & & \Big\downarrow \varphi \\ \Omega & \xrightarrow{\subset} & \mathbb{C}^n \end{array}$$

X is connected and the diagram commutes so $\varphi \circ \rho = Id$ and $\rho(\Omega) \neq X$.
We are assuming that φ is a local homeomorphism. Pick η boundary point of $\rho(\eta)$ in X. We may assume that $\rho(\eta) = \zeta_k$. Get ψ local branch of φ^{-1} with $\psi(\zeta_k) = \eta$, for n large $\Rightarrow \psi : B(\eta, \frac{1}{n}) \longrightarrow U \subset X$ open.

$$\varphi(U \cap \rho(\Omega)) = \{z \in B(\eta, \frac{1}{n}) \cap \Omega \mid \rho(z) = \psi(z)\}$$

This is open since φ is an open map. Since X is a Riemann domain $\Rightarrow X$ is Hausdorff $\Rightarrow \varphi(U \cap \rho(\Omega))$ is closed in $B(\eta, \frac{1}{n}) \cap \Omega$. Hence it is a union of components of $B(\eta, \frac{1}{n}) \cap \Omega$. Note that $\varphi(U \cap \rho(\Omega)) \neq \emptyset$, so there is at least one component. Conclude: h does not extend to X. □

Lecture 18. February 18, 2009

Definition 0.97. Ω is a **domain of holomorphy** if Ω satisfies conditions $(1) - (5)$ in the previous theorem.

Definition 0.98. Ω is **$\mathbb{C}$-linearly convex** if $\mathbb{C}^n \backslash \Omega$ is a union of complex affine hyperplanes.

Definition 0.99. A **complex affine hyperplane** in $\mathbb{C}^n$ is a translate of a $\mathbb{C}$-subspace of dimension $n - 1$.

Definition 0.100. Ω is **$\mathbb{C}$-convex** if each intersection of Ω with a complex affine line is connected and simply connected.

Fact:

- Ω is $\mathbb{C}$-convex $\Rightarrow \Omega$ is $\mathbb{C}$-linearly convex
- If $\Omega \subseteq \mathbb{C}^n$ is $\mathbb{C}$-linearly convex, $b\Omega$ is C^1, and $n \geq 2 \Rightarrow \Omega$ is $\mathbb{C}$-convex

Proposition 0.101. *Ω is convex, then Ω is $\mathbb{C}$-linearly convex.*

Proof. Take a point $p \in \mathbb{C}^n \setminus \Omega \Rightarrow$ (see Hormander's Notions of Convexity for proof) $\exists$ a real affine hyperplane H through p such that Ω lie on one side of H. H contains a complex affine hyperplane H' through p. Since $H' \subset H$, H' is also disjoint from Ω. □

Proposition 0.102. *If Ω is $\mathbb{C}$-linearly convex, then Ω is a domain of holomorphy*

Proof. Take $p \in b\Omega \Rightarrow p \in H' = \{z \mid l(z) = c\}$, where $l : \mathbb{C}^n \longrightarrow \mathbb{C}$ is $\mathbb{C}$-linear and c is a constant. Then $f(z) = \frac{1}{l(z)-c}$ is singular at p "from all directions." $\Rightarrow$ there is no auto extension of f. □

Corollary 0.103. *If Ω is convex, then Ω is a domain of holomorphy.*

Example 0.104. Suppose that $\Omega = \{(z, w) \in \mathbb{C}^2 \mid z \neq 0\}$. Ω is not convex, it is $\mathbb{C}$-linearly convex, and it is not $\mathbb{C}$-convex.

Lecture 19. February 20, 2009

Let Ω be a Reinhardt domain.

$\Omega$is a domain of holomorphy $\Rightarrow^{\text{cond. } 5}$ Ω is a domain of convergence for some $\sum c_\alpha z^\alpha$

$\Rightarrow$ i) Ω is logarithmically convex and ii) $\forall j, \Omega \cap \{z_j = 0\} = \emptyset$ or $Proj_{\{z_j=0\}}\Omega \subset \Omega$

$\Rightarrow$ Ω is a domain of holomorphy

Corollary 0.105. *If Ω is Reinhardt, then Ω is a domain of convergence $\Leftrightarrow$ Ω satisfies i) and ii).*

Proof. We want to check $\Leftarrow$ since we already have shown the $\Rightarrow$ implication. Our goal is to show that there is a function that cannot extend beyond Ω. When $n = 2$: if there is a point on Ω that has a tangent line with an irrational slope when we are looking at $\log(\Omega)$, we can push out and tilt slight at that point to get a rational slope $\frac{m}{n}$, where m, n are rational. Then we can use the function: $\frac{1}{z_1^m z_2^n - e^{c+i\theta}}$, where $\log |z_1^m z_2^n| = c$. □

One more class of examples:
Given $U \subset \mathbb{C}^n$ open, $f = (f_1, \ldots, f_m) \longrightarrow \mathbb{C}^m$, and $\Delta^m \subset \mathbb{C}^m$ the unit polydisk. Then:

$f^{-1}(\Delta^m) = \{z \in U \mid ||f(z_j)| < 1 \forall j\}$ is an analytic polyhedron $\Leftrightarrow f^{-1}(\Delta^m)$ has compact closure in U

Proposition 0.106. *Analytic polyhedra are domains of holomorphy.*

Proof. Look at $\frac{1}{f_j(z) - e^{i\theta}}$ and condition (5) in the "defniition" of domain of holomorphy. □

Proposition 0.107. *If $\{\Omega_\alpha\}_{\alpha \in A}$ are domains of holomorphy, then $\Omega = Int(\cap_\alpha \Omega_\alpha$ is a domain of holomorphy.*

Proof. Use condition (2) in the "definition" of domain of holomorphy.

$$\begin{aligned} dist_{L^\infty}(\hat{K}_{Holo(\Omega)}, b\Omega) &= \inf_\alpha dist_{L^\infty}(\hat{K}_{Holo(\Omega)}, b\Omega_\alpha) \\ &\geq \inf_\alpha dist_{L^infty}(\hat{K}_{Holo(\Omega_\alpha)}, b\Omega_\alpha) \\ &= \inf dist_{L^\infty}(K, b\Omega_\alpha) \\ &= dist_{L^\infty}(K, b\Omega) \end{aligned}$$

□

Proposition 0.108. *If Ω_1 and Ω_2 are domains of holomorphy, then $\Omega_1 \times \Omega_2$ is a domain of holomorphy.*

Proof. Use condition (3). Let compact $K \subset \Omega_1 \times \Omega_2$. Then $K \subset K_1 \times K_2$ where K_1, K_2 are compact. Then

$$hatk_{Holo(\Omega)} \subset \widehat{K_1 \times K}_{2Holo(\Omega)} \subset \hat{K}_{1Holo(\Omega_1)} \times \hat{K}_{2Holo(\Omega_2)}$$

□

Despite these results, unless we are lucky, it is difficult to determine if a subset is a domain of holomorphy.

Example 0.109. Let $\Omega = \{(z, w) \mid |z| > 1, |z-3| > 1, Re(e^{i(\log|z| + \sqrt{2}\log|z-3|)})w > 0$. This can also be described as: $|\frac{1}{z}| < 1, |\frac{1}{z-3}| < 1, |\frac{z^i(z-3)^{\sqrt{2}i}w-1}{z^i(z-3)^{\sqrt{2}i}w+1}| < 1$. Note that the imaginary numbers makes this a multi-valued function. So Ω is a "local analytic polyhedra."
Challenge: Show that Ω is a domain of holomorphy.

Lecture 20. March 2, 2009

$H^n_\epsilon = \{(z_1,\dots,z_n) \mid 1-\epsilon < |z_1| < 1, |z_j| < 1 \text{ for } j>1\} \cup \{(z_1,\dots,z_n) \mid |z_1| < 1, |z_j| < \epsilon \text{ for } j > 1\}$
Holomorphic functions on H^n_ϵ extend to Δ^n.
Suppose that $\psi : \Delta^n \longrightarrow \mathbb{C}^n$ is bi-holomorphic (i.e. injective, $\psi(\Delta^n)$ is open, and ψ^{-1} is holomorphic). $\psi(H^n_\epsilon) \subset \Omega \subset^{\text{open}} \mathbb{C}^n$. Then holomorphic functions on Ω extend to:
$X = \Omega \sqcup \psi(\Delta^n)/\sim$ where $\sim$ is given by glueing along components of $\Omega \cap \psi(\Delta^n)$ containing $\psi(H^n_\epsilon)$
Ω is a domain of holomorphy (NO AUTO EXTENSION) $\Rightarrow$ ψ is as above $\Rightarrow$ $\psi(\Delta^n) \subset \Omega$ (HARTOGS FIGURE PROPERTY).
Suppose that $\overline{\Delta} \times \{0\} \times \cdots \times \{0\} \subset \Omega$. Pick $\overrightarrow{v} \in b\Omega$ closet to 0. Suppose that $\overrightarrow{v}$ independent of $\overrightarrow{e_1} = (1,0,\dots,0)$. Let $f : \overline{\Delta} \longrightarrow \mathbb{C} \setminus \{0\}$ be holomorphic. Choose a basis $\overrightarrow{e_1}, \frac{\overrightarrow{v}}{||\overrightarrow{v}||}, \overrightarrow{b_3}, \dots, \overrightarrow{b_n}$ with $||b_j|| \le \frac{\delta}{100n}$, where $\delta > 0$ is arbitrary.

$$\psi : \Delta^n \longrightarrow \mathbb{C}^n \text{ where } (z_1,\dots,z_n) \mapsto z_1\overrightarrow{e_1} + f(z_1)(z_2\frac{\overrightarrow{v}}{||\overrightarrow{v}||} + z_3\overrightarrow{b_3} + \cdots + z_n\overrightarrow{b_n}$$

$\dagger\psi(H^n_\epsilon) \subset \Omega$ for $\epsilon < \epsilon_0$ if $(1+\delta)|f(z)| \le dist((z,0,\dots,0), b\Omega)$ for $|z| = 1$.
$\ddagger\psi(\Delta^n) \not\subseteq \Omega$ if $|f(0)| > dist(\overrightarrow{0}, b\Omega) = ||\overrightarrow{z}||$

Proof. Look at $\psi(0,z_2,0,\dots,0)$ with $\overrightarrow{z} = (0,z_2,0,\dots,0)$ and $|z_2|$ barely less than 1. □

What's a useful for f? Let u be continuous on $\overline{\Delta}$ and harmonic on Δ.

$$u(z) = \log(dist((z,0,\dots,0), b\Omega)) \text{ for } |z| = 1 \textit{(see Ahlfors 4.6.4)}$$

Let v be a harmonic conjugate for u. Then $f = \frac{e^{u+iv}}{1+\delta}$ satisfies the hypothesis †. Then Ω satisfies the Hartogs figure property $\Rightarrow$ ‡ fails $\Rightarrow$ $|f(0)| = \frac{e^{u(0)}}{1+\delta} \le dist(\overrightarrow{0}, b\Omega) \forall \delta > 0$, since δ was arbitrary $\Rightarrow$ $e^{u(0)} \le dist(\overrightarrow{0}, b\Omega)$ $\Rightarrow$ $u(0) \le dist(\overrightarrow{0}, b\Omega)$ and so $u(0) = \frac{1}{2\pi}\int_0^{2\pi} \log dist((e^{i\theta},0,\dots,0), b\Omega)d\theta \Rightarrow -\log dist(\overrightarrow{0}, b\Omega) \le avg_\theta(-\log dist(e^{i\theta}\overrightarrow{e_1}, b\Omega)$ $(***)$ (note: the integral at the origin is the average of the boundary values).
What if $\overrightarrow{z}$ is a multiple of $\overrightarrow{e_1}$? (i.e. it is on the z_1-axis).

Assume $n = 1$.

$$\begin{aligned} -\log dist(0, b\Omega) &= \max_{z \in b\Omega}(-\log|z|) \\ &=^{\text{MVT}} \max_{z\in b\Omega} avg_\theta(-\log|z - e^{i\theta}|) \\ &\le avg_\theta \max_{z \in b\Omega}(-\log|z - e^{i\theta}|) \\ &= avg(-\log dist(e^{i\theta}, b\Omega)) \end{aligned}$$

Assume $n > 1$.

$$\begin{aligned} -\log dist(\overrightarrow{0}, b\Omega) &\le avg_\theta(-\log dist(e^{i\theta}\overrightarrow{e_1}, b\Omega \cap (\mathbb{C} \cap \{0\}^{n-1})) \\ &\le avg_\theta(-\log dist(e^{i\theta}\overrightarrow{e_1}, b\Omega)) \end{aligned}$$

Proposition 0.110. *If Ω satisfies the Hartog's Figure property and $\alpha : \overline{\Delta} \longrightarrow \Omega$ is affine, then* $-\log dist(\alpha(0), b\Omega) \le avg_\theta(-\log dist(\alpha(e^{i\theta}), b\Omega))$

Proof. Reduce to the previous case using unitary maps, translations, and dilations. □

The function $-\log$(distance to the boundary) has the property that it sub-averages along any disk. In otherwords, $-\log dist(;b\Omega)$ is **plurisubharmonic**. So now we will look into more plurisubharmonic functions next.

Lecture 21. March 4, 2009

Definition 0.111. Let $\Omega \subseteq \mathbb{C}^n$ be open and $u : \Omega \longrightarrow \mathbb{R}$ continuous. Then u is **plurisubharmonic** if $u(\alpha(0)) \leq avg_\theta u(\alpha(e^{i\theta}))\forall \alpha : \overline{\Delta} \longrightarrow \Omega$ where α is $\mathbb{C}$-affine.

Note: We are assuming that u is continuous. If we don't make this assumption we need to make other assumptions (for discontinuous plurisubharmonic functions). **Facts:**

- Plurisubharmonic functions are closed under addition
- $u(z) = ||z||^2$ is a continuous plurisubharmonic function

Definition 0.112. An **exhaustion** function for Ω is a continuous function $u : \Omega \longrightarrow \mathbb{R}$ such that $\{z \in \Omega \mid u(z) \leq c\}$ is compact $\forall c \in \mathbb{R}$. (i.e. $u(z) \to \infty$ as $z \to b\Omega$ or $||z|| \to \infty$ within Ω).

Domain of Holomorphy $\Rightarrow$
Hartog's figure property $\Rightarrow$
$-\log dist(z, b\Omega)$ is continuous and plurisubharmonic $\Rightarrow$
Ω has a plurisubharmonic exhaustion function

Of last implication. $u(z) = -\log dist(z, b\Omega) + ||z||^2$ □

Eventually we will show that all of these implications go both directions.

Information on Plurisubharmonic Functions:
When $n = 1$, plurisubharmonic functions are called **subharmonic**.

Theorem 0.113. *If $\Omega \subset \mathbb{C}, u : \Omega \longrightarrow \mathbb{R}$ is C^2, then u is subharmonic $\Leftrightarrow \Delta u \geq 0$.*

Proof. Without loss of generality, assume that $0 \in \Omega$ and let $M(r) = avg_{|z|=r}u$. $M(r) \to u(0)$ as r decreases to 0.

$$\begin{aligned} M'(r) &= \frac{d}{dr}\big(\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta\big) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial u}{\partial r}(re^{i\theta})d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} (\frac{\partial u}{\partial x}\frac{x}{r} + \frac{\partial u}{\partial y}\frac{y}{r})(re^{i\theta})d\theta \\ &= \frac{1}{2\pi r}\int_{|z|=r} (\frac{\partial u}{\partial x}dy - \frac{\partial u}{\partial y}dx) \\ &= \frac{1}{2\pi r}\int\int_{|z|\leq r} \Delta u dA \end{aligned}$$

We used polar coordinate to go from line 2 to 3: $x = r\cos\theta, y = r\sin\theta, \frac{\partial x}{\partial r} = \frac{x}{r}$ and $\frac{\partial y}{\partial r} = \frac{y}{r}$. Also, on $|z| = r, dx = -yd\theta$ and $dy = xd\theta$. The last step is by Green's theorem.

$$\Delta u \geq 0 \Rightarrow M(r) \text{ is increasing in } r, M(r) \geq u(0)$$

$$\Delta u(0) < 0 \Rightarrow M(r) \text{ is strictly decreasing for small } r, M(\epsilon) < u(0)$$

Since 0 is not a special point, we can translate any point to 0 and get the same result. □

RECALL: $\Delta u = 4\frac{\partial^2 u}{\partial z \partial \bar{z}}$. Assume that u is C^2.

$$u \text{ is plurisubharmonic } \Leftrightarrow \frac{\partial^2 u}{\partial z \partial \bar{z}}(\vec{p} + a_1 z\vec{e_1} + \cdots + a_n z\vec{e_n}|_{z=0} \geq 0 \forall \vec{p}, \vec{a}$$

Claim: $\frac{\partial^2 u}{\partial z \partial \bar{z}}(\vec{p} + a_1 z\vec{e_1} + \cdots + a_n z\vec{e_n}|_{z=0} = \sum_{j,k} \frac{\partial^2 u}{\partial z_i \partial \overline{z_k}}(\vec{p}) a_j \overline{a_k}$
Justification:

$$\begin{aligned}
(u \circ \alpha)' &= (u' \circ \alpha)\alpha' \\
&= (\partial u \circ \alpha + \overline{\partial} u \circ \alpha)(\partial \alpha + \overline{\partial}\alpha) \\
&\text{this is a combination of } \mathbb{C}\text{-linear terms and two are conjugate linear} \\
\partial(u \circ \alpha) &= (\partial u \circ \alpha)\partial\alpha + (\overline{\partial} u \circ \alpha)\overline{\partial}\alpha \\
\overline{\partial}(u \circ \alpha) &= (\overline{\partial} u \circ \alpha)\partial\alpha + (\partial u \circ \alpha)\overline{\partial}\alpha \\
&\text{in the previous two lines the second term disappears since } \alpha \text{ is } \mathbb{C} \text{ affine} \\
\frac{\partial(u \circ \alpha}{\partial \bar{z}} &= \sum_k \frac{\partial u}{\partial \overline{z_k}} \overline{a_k} \\
\frac{\partial^2 (u \circ \alpha)}{\partial z \partial \bar{z}} &= \sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}} \overline{a_k} a_j
\end{aligned}$$

The term $\frac{\partial^2 u}{\partial z_j \partial \overline{z_k}}$ is the $\mathbb{C}$-Hessian of u.
If u is C^2, then u is plurisubharmonic if and only if all of its $\mathbb{C}$-Hessians are atleast 0.

0.0.1. *Digression on Hessians.* Let $\Omega \subseteq \mathbb{R}^n$ be open and $u : \Omega \longrightarrow \mathbb{R}$ be C^2. The Hessian of u at p is $\frac{\partial^2 u}{\partial x_j \partial x_k}(p)$.

$$u(x) = u(p) + u'(p)(x-p) + \frac{1}{2}(x-p)^{tr}(\text{Hessian of } u \text{ at } p)(x-p) + (\text{higher-order terms})$$

Note that $(x-p)$ is a column vector.
What happens to the Hessian under a change of variable?
Special Case: $\mathbb{R} \xrightarrow{\varphi} \mathbb{R} \xrightarrow{u} \mathbb{R}$

$$\begin{aligned}
(u \circ \varphi)' &= (u' \circ \varphi)\varphi' \\
(u \circ \varphi)'' &= (u'' \circ \varphi)(\varphi')^2 + (u' \circ \varphi)\varphi''
\end{aligned}$$

In the second line we have a term that does not involve u'' at all, which we do not like. In general it is better to view u', u'' as a package (called a "2-jet"). We do get a good translation law for u'' by itself if $\varphi'' = 0$ (e.g. φ is affine) or if $u' = 0$ (i.e. at a critical point of u).

Exercise 0.114. Let $u(x) = x^2$. This is a convex function, show how it can be made into a concave function by a change of variable except at $x = 0$.(The point $x = 0$ is an exception because it is a critical point of u).

Lecture 22. March 6, 2009

Today's Assumptions:
All functions are assume to be C^2 and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ means that $f : \Omega \longrightarrow \mathbb{R}^m$ where $\Omega \subset \mathbb{R}^n$ is open.

$$\mathbb{R}^s \xrightarrow{\varphi} \mathbb{R}^n \xrightarrow{u} \mathbb{R}$$

The Hessian of u at a is: $\frac{\partial^2 u}{\partial x_j \partial x_k}(a)$.
The Hessian of $u \circ \varphi$ at p is ?
Method 1: $(u \circ \varphi)_{jk} = \dots$
Method 2: $\varphi(x) + \varphi(p) + \varphi'(p)(x-p) + \frac{1}{2}(x-p)^{\text{tr}}(\text{Hessian of } \varphi \text{ at } p)(x-p) + $(Higher-order

terms)

$$\begin{aligned} u(\varphi(x)) &= u(\varphi(p)) + u'(\varphi(p))\varphi'(p)(x-p) \\ &+ \frac{1}{2}u'(\varphi(p))((x--p)^{\text{tr}}(\text{Hessian of } \varphi \text{ at } p)(x-p) \\ &+ \frac{1}{2}(x-p)^{\text{tr}}(\varphi'(p))^{\text{tr}}(\text{Hessian of } u \text{ at } \varphi(p))\varphi'(p)(x-p) \\ &+ (\text{Higher-Order terms}) \end{aligned}$$

Note that tr stands for transpose.

$v^{\text{tr}}(\text{Hessian of } u \circ \varphi \text{ at } p)v = v^{\text{tr}}\varphi'(p)^{\text{tr}}(\text{Hessian of } u \text{ at } p)\varphi'(p)v + u'(\varphi(p))(v^{\text{tr}}(\text{Hessian of } \varphi \text{ at } p)v)$

Note that the terms are row or column vectors and not matrices. We call the last term, $u'(\varphi(p))(v^{\text{tr}}(\text{Hessian of } \varphi \text{ at } p)v)$, the *error term* since we prefer to work with the other terms. We like the second term since it corresponds to quadratic forms on the tangent space. The error term disappears if:

(1) φ is affine (then the Hessian of φ is zero)
(2) $u'(\varphi(p)) =$, i.e. $\varphi(p)$ is a critical point of u

Corollary 0.115. *The signature of the Hessian (i.e. the number of positive, negative, and null directions) is diffeomorphism invariant at critical points.*

Definition 0.116. A function u is convex if $u(tp + (1-t)q) \leq tu(p) + (1-t)u(q), \forall p, q, 0 \leq t \leq 1$.

Fact: $u : \mathbb{R} \longrightarrow \mathbb{R}$ is convex $\Leftrightarrow u'' \geq 0$.

$$\begin{aligned} u : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ is convex } &\Leftrightarrow u \text{ is convex along each line} \\ &\Leftrightarrow u''(x,v) = v^{\text{tr}}(\text{Hessian of } u \text{ at } p) \geq 0, \forall x, v \\ &\Leftrightarrow \text{all Hessians are atleast zero} \end{aligned}$$

Corollary 0.117. *If u is convex and φ is affine, then $u \circ \varphi$ is convex.*

Now consider $u : W \longrightarrow \mathbb{R}$, where $W \subset \mathbb{C}^n$ is open.
The $\mathbb{R}$-Hessian is: $\sum \frac{\partial^2 u}{\partial x_j \partial x_k} + \dots$
Converting this into complex variables ($x_j = \frac{z_j + \overline{z_j}}{2}$), we get:

$$\sum \frac{\partial^2 u}{\partial x_j \partial x_k} + \dots = \sum \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}} z_j \overline{z_k} + \sum \frac{\partial^2 u}{\partial z_j \partial z_k} z_j z_k + \sum \frac{\partial^2}{\partial \overline{z_j z_k}} \overline{z_j z_k}$$

The first term on the right is Hermitian, and the other two are anit-Hermitian. So $\mathbb{C}$-Hessian of u is the Hermitian part of the real Hessian.

$$\begin{aligned} u \text{ is plurisubharmonic} &\Leftrightarrow \text{All } \mathbb{C}\text{-Hessians are at least } 0 \\ &\Leftrightarrow n_-^{\mathbb{R}}(\mathbb{R} - \text{Hessian}) \leq n_-^{\mathbb{C}}(\mathbb{C}\text{-Hessian}) + n = n \end{aligned}$$

Also, $u(x_1, \dots, x_n)$ is a plurisubharmonic function of $(x_1 + iy_1, \dots, x_n + iy_n) \Leftrightarrow u$ is convex.

Now consider $\mathbb{C}^s \xrightarrow[\text{holo.}]{\varphi} \mathbb{C}^n \xrightarrow{u} \mathbb{R}$.

The $\mathbb{R}$-Hessian of φ is anti-Hermintian, so the $\mathbb{C}$-Hessian of $u \circ \varphi$ at $p = (\varphi')^{\text{tr}}(\mathbb{C}\text{-Hessian of } u)\varphi'$.

Corollary 0.118. *If u is plurisubharmonic and φ is holomorphic, then $u \circ \varphi$ is plurisubharmonic.*

Corollary 0.119. *For C^2 functions, plurisubharmonicity is biholomorphic-invariant*

Lecture 23. March 9, 2009

Again, we want to assume that all functions are C^2.

Proposition 0.120. *Given $\mathbb{R}^s \xrightarrow{\varphi} \mathbb{R}^n \xrightarrow{u} \mathbb{R}$, where u is convex and non-decreasing in each variable and the φ_j are all convex, then $u \circ \varphi$ is convex.*

Proof. We need to look at this equation:

$v^{\text{tr}}(\text{Hessian of } u\circ\varphi \text{ at } p)v = v^{\text{tr}}\varphi'(p)^{\text{tr}}(\text{Hessian of } u \text{ at } p)\varphi'(p)v + u'(\varphi(p))(v^{\text{tr}}(\text{Hessian of } \varphi \text{ at } p)v)$

All of the entries of the vector $u'(\varphi(p))$ are non-negative since we assumed that it was non-decreasing in each variable and all the entries of (Hessian of φ at p) are non-negative. Also the second term has non-negative entries so $u \circ \varphi$ is convex. □

Proposition 0.121. *Given $\mathbb{C}^s \xrightarrow{\varphi} \mathbb{R}^n \xrightarrow{u} \mathbb{R}$, where u is convex and non-decreasing in each variable and the φ_j are all plurisubharmonic, then $u \circ \varphi$ is plurisubharmonic.*

Proof. Take the Hermittian part of each term in:

$v^{\text{tr}}(\text{Hessian of } u\circ\varphi \text{ at } p)v = v^{\text{tr}}\varphi'(p)^{\text{tr}}(\text{Hessian of } u \text{ at } p)\varphi'(p)v + u'(\varphi(p))(v^{\text{tr}}(\text{Hessian of } \varphi \text{ at } p)v)$

Use lemma: $Q \geq 0 \Rightarrow Q^{\text{Herm.}} \geq 0$ since

$$Q^{\text{Herm.}}(v, v) = \frac{Q(v, v) + Q(Jv, Jv)}{2}$$

□

In the two previous propositions, the same result holds if we restrict φ to open subsets of its given domain and range.

Corollary 0.122. *If u is plurisubharmonic, then e^u is plurisubharmonic.*

Corollary 0.123. *If $u_j > 0$ is plurisubharmonic, then $(u_1^p + \cdots + u_n^p)^{\frac{1}{p}}$ is plurisubharmonic for $p \geq 1$.*

We would like to get versions of these results without having to assume that the functions are all C^2.
Idea: Approximate convex/plurisubharmonic functions by smooth convex/plurisubharmonic functions.
Recall: Let $\Omega \subset \mathbb{C}^n$ be open. Then $u : \Omega \longrightarrow \mathbb{R}$ is continuous and plurisubharmonic $\Leftrightarrow$ $U(\alpha(0)) \leq avg_\theta u(\alpha(e^{i\theta})), \forall\mathbb{C}$-affine maps $\alpha : \overline{\Delta} \longrightarrow \Omega$.
Note: If we have a sequence of functions u_j that are continuous and plurisubharmonic such that $u_j \to u$ almost uniformly, then u is plurisubharmonic.
Pick $\eta \in C^\infty(\mathbb{C}^n)$ such that:

(1) $\eta(z) = 0$ for $||z|| > \frac{1}{2}$
(2) $\eta \geq 0$
(3) $\int \eta = 1$

$$\begin{aligned} u_\epsilon(z) &\equiv \int_{\mathbb{C}^n} u(z - \epsilon\zeta)\eta(\zeta)dV_\zeta \\ &= \int u(\zeta)\eta(\frac{z-\zeta}{\epsilon}\frac{dV_\zeta}{\epsilon^{2n}} \\ &= u(z) + \int (-u(z) + u(\zeta))\eta(\frac{z-\zeta}{\epsilon}\frac{dV_\zeta}{\epsilon^{2n}} \end{aligned}$$

Exercise 0.124. u_ϵ is C^∞ on $\Omega_\epsilon = \{z \in \Omega \mid dist(z, b\Omega) > \epsilon\}$.

Exercise 0.125. $u_\epsilon \to u$ almost uniformly as $\epsilon \to 0$.

Exercise 0.126. If u is plurisubharmonic, then u_ϵ is plurisubharmonic (use Fubini's theorem). If we start with η in $\mathbb{R}^n$ instead of $\mathbb{C}^n$, we get the result that u is convex $\Rightarrow u_\epsilon$ convex.

Assume the results from all of these exercises. Then:

$$u \text{ is plurisubharmonic } \Leftrightarrow u_\epsilon \text{ is plurisubharmonic for } 0 < \epsilon < \epsilon_0$$

Corollary 0.127. *Locally plurisubharmonic functions are plurisubharmonic.*

From the definition of plurisubharmonic, it was unclear that this was a local condition. However, from the results above that we can approximate a plurisubharmonic u by u_ϵ, which are defined locally and are plurisubharmonic, it is now clear that plurisubharmonic is a local condition.

Corollary 0.128. *If u is continuous, plurisubharmonic and φ is holomorphic, then $u \circ \varphi$ is continuous and plurisubharmonic.*

Proof. $u \circ \varphi = \lim u_\epsilon \circ \varphi$ □

Corollary 0.129. *If u is plurisubharmonic, then u is subaveraging along every holomorphic disk.*

Note: we already know that u is subaveraging along every affine disk, but the above result extends this further to every holomorphic disk.

Corollary 0.130. *Plurisubharmonicity is a biholomorphic invariant notion.*

Corollary 0.131. *Let $\Omega \subseteq \mathbb{C}^n$ be open. $\Omega \xrightarrow{\varphi} \mathbb{R}^n$, where each component is plurisubharmonic and $\mathbb{R}^n \xrightarrow{u} \mathbb{R}$ is convex and non-decreasing in each variable, then $u \circ \varphi$ is plurisubharmonic.*

Proof. $u \circ \varphi = \lim_{\epsilon \to 0} u_\epsilon \circ \varphi_\epsilon$.. □

Examples:

(1) If u is plurisubharmonic, then e^u is plurisubharmonic.
(2) If $u < \alpha$ is plurisubharmonic, then $\frac{1}{\alpha - u}$ is plurisubharmonic.
(3) If $\{u_j\}_{j=1}^n$ are plurisubharmonic, then $u_1 + \cdots + u_n$ is plurisubharmonic.
(4) If $\{u_j\}_{j=1}^n$ are plurisubharmonic, then $\max\{u_1, \ldots, u_n\}$ is plurisubharmonic.
(5) If u_1, u_2 are plurisubharmonic, then $\log(e^{u_1} + e^{u_2})$ is plurisubharmonic.

    Proof. Check that the Hessian is $\frac{1}{(e^{u_1}+e^{u_2})^2}\begin{pmatrix} e^{iu_1} & e^{u_1+u_2} \\ e^{u_1+u_2} & e^{iu_2} \end{pmatrix}$ □

(6) If $\{u_j\}_{j=1}^n$ are plurisubharmonic, then $\log(e^{u_1} + \cdots + e^{u_n})$ is plurisubharmonic.

    Proof. Use a larger Hessian or induction with the previous example. □

(7) If $\{u_j\}_{j=1}^n$ are plurisubharmonic, $u_j \geq 0$, and $p > 1$, then $(u_1^p + \cdots + u_n^p)^{\frac{1}{p}}$ is plurisubharmonic.

Remark 0.132. Plurisubharmonic functions are much more flexible than holomorphic functions. If we want to construct a holomorphic function satisfying a set of conditions, it can be very difficult because of the unique continuation principle (i.e. we cannot hold the function fixed in one area while varying it in another and still ensure that the function remains holomorphic. However, we can do this with plurisubharmonic functions. We will see how to get holomorphic functions from plurisubharmonic functions (the reason why we are in fact studying plurisubharmonic functions).

Lecture 24. March 11, 2009

Recall:

Domain of Holomorphy $\Leftrightarrow^{\text{def.}}$ No auto extension
$\Leftrightarrow$
$\Leftrightarrow$ Holomorphic Convexity
$\Leftrightarrow$ Discrete Interpolation
$\Leftrightarrow$ Domain of Existence
$\Rightarrow$ Hartog's Figure Property
Pseudoconvex Domain $\Leftrightarrow^{\text{def.}}$ Hartog's Figure Property
$\Leftrightarrow -\log dist(z, b\Omega)$ is plurisubharmonic
$\Leftrightarrow$ plurisubharmonic exhaustion function
$\Leftrightarrow$ plurisubharmonic convexity - *by today*
$\Leftrightarrow$ Kontinuitatsatz - *by today*
$\Leftrightarrow$ Hartog's Figure Property - *by today*
$\Rightarrow$ Domain of holomorphy - "solution of the Levi problem" - *by today*
Domain of Holomorphy $\Leftrightarrow$ Pseudoconvex Domain

$$\hat{K}_{\text{plsh}(\Omega)} \equiv \{z \in \Omega \mid u(z) \leq \max_k u, \forall u \text{ continuous, plurisubharmonic on } \Omega\}$$

Proposition 0.133. *If Ω has a plurisubharmonic function v, then $\hat{K}_{plsh(\Omega)}$ is compact for all compact $K \subset \Omega$.*

Proof. $\hat{K}_{\text{plsh}(\Omega)} \subset \{z \in \Omega \mid v(z) \leq \max_K v\}$ is relatively closed in Ω so it is compact. □

$\overline{\Delta} \xrightarrow[\text{cont.}]{\alpha} \Omega \xrightarrow[\text{plsh}]{u} \mathbb{R}$, where α is holomorphic on Δ.
$\Rightarrow u \circ \alpha$ is continuous on $\overline{\Delta}$ and subharmonic on Δ
$\Rightarrow u(\alpha(0)) \leq avg_\theta u(\alpha(e^{i\theta})) \leq \max_{\alpha(b\Delta)} u$
Composing with the auto of Δ, $\max_{\alpha(\overline{\Delta})} = \max_{\alpha(b\Delta)} u$. So $\alpha(\overline{\Delta}) \subset \alpha(\hat{b\Delta}))_{\text{plsh}(\Omega)}$.

Kontinuitatsatz or Disk Property
Suppose that Ω is plurisubharmonic convex. Then:
$\alpha_z : \overline{\Delta} \longrightarrow \Omega$ is continuous on $\overline{\Delta}$, is holomorphic on Δ
$\overline{\cup_v \alpha_v(b\Delta)}$ is compact in $\Omega \Rightarrow \overline{\cup_v \alpha_v(\overline{\Delta})}$ is compact in Ω
Also, Kontinuitatsatz implies the Hartog's figurre property. In the 1950's it was proven (difficult proof) that a pseudoconvex domain is a domain of holomorphy. Hence domains of holomorphy and pseudoconvex domains are equivalent.

Proposition 0.134. *Suppose Ω_1, Ω_2 are pseudoconvex domains, then $\Omega_1 \cap \Omega_2$ is a pseudoconex domain.*

Proof. $-\log dist(z, b(\Omega_1 \cap \Omega_2)) = \max\{-\log dist(z, b\Omega_1), -\log dist(z, b\Omega_2)\}$ □

Proposition 0.135. *Ω is pseudoconvex $\Leftrightarrow \forall p \in b\Omega, \exists \epsilon > 0$ such that $\Omega \cap B(p, \epsilon)$ is pseudoconvex.*

Proof. $(\Rightarrow)$ use the above proposition. □

Assuming Ω is bounded

$$-\log dist(z, b\Omega) = -\log dist(z, \Omega \cap B(p,\epsilon)) \text{ for } z \in \Omega \cap B(p, \frac{\epsilon}{2})$$

Cover $b\Omega$ with finitely many balls of radius $\frac{\epsilon}{2}$.
$-\log dist(z, b\Omega)$ is plurisubharmonic on $\Omega \setminus K$, where K compact.
$u(z) = \max\{-\log dist(z, b\Omega), -\log dist(K, b\Omega) + 1\}$ is a plurisubharmonic exhaustion function:
For any point in K, $-\log dist(K, b\Omega) + 1$ dominates the other value and it is constant, hence plurisubharmonic. Use the fact that plurisubharmonicity is local to show that $u(z)$ is plurisubharmonic.
Assuming Ω is unbounded

$$\begin{aligned} \Omega \text{ is locally pseudoconvex } &\Rightarrow \Omega \cap B(0, M) \text{ is pseudoconvex} \\ &\Rightarrow -\log dist(z, b(\Omega \cap B(0, M)) \text{ is plurisubharmonic} \\ &\Rightarrow -\log dist(z, b\Omega) \text{ is plurisubharmonic since plurisubharmonicity is local} \end{aligned}$$

Lecture 25. March 13, 2009

Let $\Omega \subset \mathbb{R}^n$ open. $\Omega + i\mathbb{R}^n \subset \mathbb{C}^n$ is called the **tube** on Ω.

Proposition 0.136. *$u \in C(\Omega)$ is plurisubharmonic on $\Omega \times i\mathbb{R}^n \Leftrightarrow u$ is convex.*

Proof. u is a function on $\Omega \times i\mathbb{R}^n$ by ignoring the terms in $i\mathbb{R}^n$
C^2 case: $\left(\frac{\partial^2 u}{\partial z_j \overline{z_k}}\right) = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right)$
General case: exercise - use convolutiosn in x variables following the previous lecture. ☐

Corollary 0.137. *Assume that $\Omega \subset \mathbb{R}^n$ is connected. Then $\Omega + i\mathbb{R}^n$ is pseudoconvex $\Leftrightarrow \Omega$ is convex.*

Proof. ($\Leftarrow$)
Omega is convex $\Rightarrow \Omega + i\mathbb{R}^n$ is convex $\Rightarrow \Omega + i\mathbb{R}^n$ is a domain of holomorphy $\Rightarrow \Omega + i\mathbb{R}^n$ is pseudoconvex.
($\Rightarrow$) $\Omega + i\mathbb{R}^n$ is pseudoconvex $\Leftrightarrow -\log dist(z, b(\Omega + i\mathbb{R}^n))$ is plurisubharmonic $\Leftrightarrow -\log dist(x, b\Omega)$ is convex $\Rightarrow$ "line segment version of Kontinuitatsatz" $\Rightarrow \Omega$ is convex. ☐

Corollary 0.138. *Suppose that $p(z) = p(x + iy)$ is a polynomial on $\mathbb{C}^n$. Then each component of $\Omega \equiv \{x \in \mathbb{R}^n \mid p(x + iy) \neq 0, \forall y \in \mathbb{R}^n\}$ is convex.*

Proof. $\Omega + i\mathbb{R}^n = \cap_y(\mathbb{C}^n \setminus \{z \mid p(z) = 0\}) + iy$ is a tube domain of holomorphy (since for every boundary point we can define a fuction that blows up on the boundary) or a disjoint union of tubes of domains of holomorphy if it is not connected. So this is a tube over a set with convex components. ☐

Example 0.139. Let $p(z_1, z_2) = z_1 z_2$. $\Omega = \{(x_1, x_2) \mid x_1 x_2 \neq 0\}$

Example 0.140. Let $p(z_1, z_2) = z_1^2 + z_2^2 - 1$. Given x is there a y such that $p(x + iy) = 0 \Leftrightarrow ||x||^2 - ||y||^2 - 1 = 0$ and $xy = 0$?

$$\Omega = \{x \in \mathbb{R}^2 \mid ||x|| < 1\}$$

Suppose $M \subset \mathbb{C}^n$ is a compact, real submanifold.
General Fact: $\exists \epsilon > 0$ such that $M_\epsilon = \{z \in \mathbb{C}^n \mid dist(x, M) < \epsilon\} \xrightarrow{\psi} \mathbb{R}$ where $\psi(z) = (dist(z, M))^2$ is smooth. Let $p \in M$ and consider the Hessian of ψ at p.

$$v^{\text{tr}}(\text{Hessian of } \psi \text{ at } p)v = \psi''(p, v,) \geq 0 \text{ since } \psi \text{ has a minimum at } p.$$

Also, $\psi(z) = ||z-p||^2$ for $z \in N_p$ $\psi''(p,z) > 0, \forall v$ normal to M at p. Claim: as we travel towards p, ψ is so small (since we took it to be the distance *square*) that it forces the second derivative to be zero for all v tangent to M at p. In particular,

$$\psi''(p,v) = 0 \Leftrightarrow v \text{ is tangent to } M \text{ at } p$$

Use: the set of null vectors for semi-definite quadratic form forms a vector space.

$$v^{\mathrm{tr}}(\mathbb{C}-\text{Hessian of } \psi \text{ at } p)v = \frac{1}{2}\{\psi''(p,v) + \psi''(p,Jv)\} \geq 0$$

$$v^{\mathrm{tr}}(\mathbb{C}-\text{Hessian of } \psi \text{ at } p)v = 0 \Leftrightarrow v \in T_pM \text{ and } Jv \in T_pM \Leftrightarrow v \in T_pM \cap JT_pM$$

$$\begin{aligned} M \text{ is totally real} &\Rightarrow \mathbb{C}-\text{Hessian of } \psi \text{ at } p \in M > 0 \\ &\Rightarrow \mathbb{C}-\text{Hessian of } \psi \text{ at } p \in M_\epsilon, \text{ for a small } \epsilon > 0 \\ &\Rightarrow \psi \text{ is strictly plurisubharmonic on } M_\epsilon, \text{ for a small } \epsilon > 0 \\ &\Rightarrow \frac{1}{\epsilon - \psi} \text{ is a plurisubharmonic exhaustion for } M_\epsilon, \text{ for a small } \epsilon > 0 \\ &\Rightarrow M_\epsilon \text{ are pseudoconvex, for a small } \epsilon > 0 \end{aligned}$$

Motivation behind the next lecture:

Example 0.141. Let $S^{n-1} = \{x \in \mathbb{R}^n \mid p(x) = 0\}$, where $p(x) = ||x||^2 - 1$. Let $\alpha : (0,1) \longrightarrow S^{n-1}$ be affine. Then $0 = (p \circ \alpha)''(t) = \alpha'(t)^{\mathrm{tr}}(\text{Hessian of } p \text{ at } \alpha(t))\alpha'(t) \Rightarrow \alpha'(t) = 0, \forall t \Rightarrow \alpha$ is constant.

Example 0.142. Let $\alpha : \Delta \longrightarrow S^{2n-1}$ be holomorphic $\Rightarrow$
$0 = \mathbb{C}$-Hessian of $p \circ \alpha$ at $z = \overline{\alpha'(z)^{\mathrm{tr}}}(\mathbb{C}\text{-Hessian of } p \text{ at } \alpha(z))\alpha'(z) \Rightarrow \alpha'(z) = 0, \forall z \Rightarrow \alpha$ is constant.

Next time: Level sets

Lecture 26. March 16, 2009

Remark 0.143. Today's lecture is presented in a different way than you can find in a standard text on this subject.

Let M be a real hypersurface in $\mathbb{C}^n$ (i.e. a real manifold of dimension $2n-1$). Everything in this lecture is atleast C^2-smooth but can be considered as C^∞ smooth.
Let ρ be the defining function for M. What happens to the Hessian of ρ if we replace ρ by $\widetilde{\rho} = \eta\rho$, where $\eta \neq 0$ and η is a smooth as necessary to meet the smoothness requirements of $\widetilde{\rho}$?
Real Case: $\widetilde{\rho}' = \rho\eta' + \eta\rho'$
$\widetilde{\rho}'' = \rho\eta'' + 2\eta'\rho' + \eta\rho''$
i.e. $\widetilde{\rho}''(x;v) = \rho(x)\eta''(x;v) + 2\eta'(x;v)\rho'(x;v) + \eta(x)\rho'(x;v)$
If $\rho(x) = 0$ (i.e. $x \in M$) and $\rho'(x;v) = 0$ (i.e. $v \in T_xM$), then

$$\widetilde{\rho}''(x;v) = \eta(x)\rho''(x;v) = \eta(x)v^{tr}(\text{Hessian of } \rho \text{ at } x)v$$

So the Hessian of $\widetilde{\rho}$ at $x|_{T_xM} = \eta(x) \cdot (\text{Hesssian of } \rho \text{ at } x|_{T_xM})$.

Define: $T_xM \xrightarrow{\mathcal{F}_x} T_x\mathbb{R}^n/T_xM$. We call $\mathcal{F}_x$ the **second fundamental form**. The first fundamental form often corresponds to the inner product on some tangent space. However, for the purposes of this lecture there is no first fundamental form because we want to be able to look at translations and rotations.

$$\rho'(x) \cdot \mathcal{F}_x(v) \equiv \rho''(x;v) = v^{tr}(\text{Hessian of } \rho \text{ at } x)v$$

Note that $\rho''(x;v)$ is a scalar, $\rho'(x)$ is a row vector, and $\mathcal{F}_x(x;v)$ is a column vector. $\mathcal{F}_x$ is a quadratic form on T_xM with values in $T_x\mathbb{R}^n/T_xM$. Suppose we have to column vectors w_1, w_2 satisfying:

$$\rho'(x)\cdot w_1 = \rho'(x)\cdot w_2 \Rightarrow \rho'(x)\cdot(w_2-w_1) = \rho'(x;w_2-w_1) = 0 \Rightarrow w_2-w_1 \in T_xM$$

Therefore them form $\mathcal{F}_x$ is well-defined.

Suppose that $\widetilde{\rho} = \eta\rho, \eta \neq 0$. Then $\widetilde{\rho}'(x)\cdot\mathcal{F}_x(v) =^? \widetilde{\rho}''(x;v)$. Since we are asumming that $x \in M$ and $v \in T_xM$, $\widetilde{\rho}''(x;v) = \eta(x)\rho''(x;v)$ (we saw this above). And $\widetilde{\rho}'(x) = \eta(x)\rho'(x)$. Hence

$$\widetilde{\rho}'(x)\cdot\mathcal{F}_x(v) = \eta(x)\rho'(x)\cdot\mathcal{F}_x(v) = \eta(x)\rho''(x;v) = \widetilde{\rho}''(x;v).$$

So:

(1) $\mathcal{F}_x$ does not depend on the choice of ρ
(2) All of this transforms naturally under affine change of coordinates
(3) M convex hypersurface $\Leftrightarrow^{\text{def}} \mathcal{F}_x \geq 0, \forall x \in M$
(4) **Fact:** If $\Omega \subset \mathbb{R}^n$ is an open, connected subset with smooth boundary. Then Ω is convex $\Leftrightarrow b\Omega$ is a convex hypersurface.

Remark 0.144. Let $\Omega \subset \mathbb{R}^n$ be an open, connected subset with smooth boundary. In order to say that $\mathcal{F}_x \geq 0$, we need to define an orientation on $T_x\mathbb{R}^n/T_xM$. Orient $T_x\mathbb{R}^n/T_xb\Omega$ so that the positive direction points out of Ω (i.e. corresponds to outside of Ω) and negative direction points into Ω

Move onto $\mathbb{C}$-Hessian: Assuming (still) that ρ is a defining function for a smooth hypersurface M.

$$\rho''_{\mathbb{C}}(x;v) \equiv \frac{1}{2}(\rho''(x;v)+\rho''(x;Jv)) = \overline{v}^{tr}(\mathbb{C}\text{-Hessian of } \rho \text{ at } x)v$$

So $\mathcal{L}_z$ is a complex quadratic form on the maximal complex tangent space. We call $\mathcal{L}_z$ the **Levi form**.
Recall: $(\rho\circ\alpha)''_{\mathbb{C}}(z;v) = \rho''_{\mathbb{C}}(\alpha(z);\alpha'(z)v)$ if α is holomorphic.
Insist that $z \in M, v \in T_zM \cap JT_zM$. Get $T_zM \cap JT_zM \xrightarrow{\mathcal{L}_z} T_z\mathbb{C}^n/T_zM$ defined by:

$$\rho'(z)\cdot\mathcal{L}_z(v) = \rho''_{\mathbb{C}}(z;v).$$

(1) This does not depend on the choice of ρ
(2) Transforms naturally under holomorphic maps (since complex Hessians transform naturally
(3) $\mathcal{L}_z$ is a Hermitian form on $T_zM \cap JT_zM$
(4) M is **Levi-pseudoconvex** $\Leftrightarrow^{\text{def}} \mathcal{L}_z \geq 0, \forall z \in M$
(5) Ω is pseudoconvex $\Leftrightarrow b\Omega$ is Levi-pseudoconvex *(we will show this)*

Remark 0.145. We are not requiring that Ω is connected when it is pseudoconvex. Instead we are assuming that each connected component is pseudoconvex.

To study: M near $z \in M$. After translation, we may assume that $z = 0$. After $\mathbb{C}$-linear change of coordinates we may assume that $T_0M = \mathbb{C}^{n-1}\times\mathbb{R}$. The implicit function theorem implies that M is locally a graph so that:

$$y_n = f(z_1,\dots,z_{n-1},x_n), \quad f(0)=0, \quad \text{and } f'(0)=0.$$

$y_n =$ quadratic terms in $(z_1,\dots,z_{n-1},x_n)+$ higher order terms.
Next time, we will use holomorphic change of variables to kill off many more terms.

Lecture 27. March 18, 2009

Theorem 0.146 ($\mathbb{R}$-inverse function theorem)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open, $F : \Omega \longrightarrow \mathbb{C}^n$ be C^k (for $k \geq 1$) and $F'(x_0)$ is invertible. Then F has a C^k inverse, G, near $F(x_0)$ and $G'(F(x)) = (F'(x))^{-1}$.*

Theorem 0.147 ($\mathbb{C}$-inverse function theorem)**.** *Let $\Omega \subseteq \mathbb{C}^n$ be open, $F : \Omega \longrightarrow \mathbb{C}^n$ holomorphic, and $F'(z_0)$ invertible. Then F has a holomorphic inverse near $F(z_0)$.*

The $\mathbb{C}$ version follows immediately from the $\mathbb{R}$ version of this theorem.

Take a point $p \in M$ a real hypersurface (in $\mathbb{R}^n$). The second fundamental form $\mathcal{F}_p : T_pM \longrightarrow T_p\mathbb{R}^n/T_pM$ is a quadratic form which is invariant under $\mathbb{R}$-affine changes in coordinates.

Definition 0.148. M is a **convex hypersurface** if all $\mathcal{F}_p \geq 0$.

Theorem 0.149. *M is a convex hypersurface $\Leftrightarrow$ M bounds a convex set.*

Definition 0.150. M is a **strongly convex hypersurface** if all $\mathcal{F}_p > 0$.

Definition 0.151. M is **strictly convex** if M is convex and M contains no line segments.

Strongly convex implies strictly convex, but not vice versa.

Example 0.152. $y = x^4$ is strictly convex, but not strongly convex.

Take a point $p \in M \subset \mathbb{C}^n$. The Levi form, $\mathcal{L}_p$, is the Hermitian part of $\mathcal{F}_p|_{T_pM\cap JT_pM}$ and this is holomorphically invariant.

Definition 0.153. M is **Levi pseudoconvex** if $\mathcal{L}_p \geq 0$ for all $p \in M$.

Assume that M is the boundary of some domain, then M is Levi pseudoconvex $\Leftrightarrow$ M bounds a pseudoconvex domain.

Definition 0.154. M is **strictly/strongly (Levi) pseudoconvex** if $\mathcal{L}_p > 0$ for all $p \in M$.

These two terms are used interchangelably, but we will prefer to use strongly Levi pseudoconvex since this definition is closer to the definition in the real case.

Idea: Use coordinate changes to simplify the geometry of M at p.
Last time: We used $\mathbb{C}$-affine change of coordinates to reduce to the case that $p = 0$ and T_pM is $\mathbb{C}^n \times \mathbb{R}$. Then M given by $y_n = f(z_1, \ldots, z_{n-1}, x_n)$, $f(0) = 0, f'(0) = 0$. Using Taylor expansion we get:

$$y_n = \sum_{1\leq j,k\leq n-1} \left(a_{jk}z_jz_k + \overline{a_{jk}z_jz_k} + b_{jk}z_j\overline{z_k} + c_jz_jx_n + \overline{c_jz_j}x_n\right) + c_nx_n^2 + (\text{higher-order terms})$$

We need this to be self-conjugate so where $b_{kj} = \overline{b_{jk}}$. We could also assume that $a_{jk} = a_{kj}$ so that the matrix would be symmetric.

$$v \in \mathbb{C}^{n-1} \times \{0\} \Rightarrow \mathcal{L}_0(v) = \sum_{1\leq j,k\leq n-1} v_{jk}z_j\overline{v_k}\frac{\partial}{\partial y_n}$$

Make <u>local</u> holomorphic change of coordinates:

$$\widetilde{z_j} = z_j \text{ for } 1 \leq j \leq n-1$$

$$\widetilde{z_n} = z_n - 2i\sum_{1\leq j,k\leq n-1} a_{jk}z_jz_k - 2i\sum_{1\leq j\leq n-1} c_jz_jz_n + i\gamma z_n^2,$$

where γ is a real number yet to be determined. This gives us:

$$\widetilde{y_n} = y_n - \sum_{1\leq j,k\leq n-1} a_{jk}z_jz_k - \sum_{1\leq j,k\leq n-1} \overline{a_{jk}z_jz_k} - \sum_{1\leq j\leq n-1} c_jz_jz_n - \sum_{1\leq j\leq n-1} \overline{c_jz_jz_n} + \gamma(x_n^2 - y_n^2)$$

On $\widetilde{M}$,

$$\widetilde{y_n} = \sum_{1\le j\le n-1} b_{jk}\widetilde{z_j}\widetilde{\overline{z_n}} + (c_n+\gamma)\widetilde{x_n}^2 + \text{(higher-order terms)}$$

So we may assume that:

$$y_n = \sum_{1\le j,k\le n-1} b_{jk}z_j\overline{z_k} + x_n^2 + \text{(higher-order terms)}$$

We may also assume that the matrix for (b_{jk}) is diagonal with 1's, -1's, and 0's (in that order) on the diagonal times a matrix J.

Lecture 28. March 20, 2009

M is a real hypersurface in $\mathbb{C}^n$:
M_+ corresponds to the inside and M_- corresponds to the outside. In particular, M_+ is the region where $y_n > \sum_{1\le j,k\le n-1} b_{jk}z_j\overline{z_k} + x_n^2 +$ (higher-order terms) and M_- is the region where $y_n < \sum_{1\le j,k\le n-1} b_{jk}z_j\overline{z_k} + x_n^2 +$ (higher-order terms).

Theorem 0.155. *M_+ is pseudoconvex $\Leftrightarrow$ M is Levi-pseudoconvex.*

Proof. There are 3 cases:

(1) **Suppose some $\mathcal{L}_pM \not\geq 0$.** Show that M_+ is not pseudoconvex.
We may assume that $b_{11} < 0$. Then $(z_1, 0, \ldots, 0) \in M_+$ for $0 < |z_1| < \delta$. For $t \in (0,\epsilon)$, $f_+ : \Delta \longrightarrow M_+$ by $z \mapsto (Sz, 0, \ldots, 0, it)$. We have a sequence of circles whose boundaries are in M_+ but whose closures are in M. Therefore Kontinuitsatz fails for M_+ and so it is not pseudoconvex.

(2) **Suppose that M is strongly pseudoconvex** (i.e. $\mathcal{L}_p > 0$). Show that M_+ is pseudoconvex.
Reduce to $y_n = \sum_{1\le j\le n+1} |z_j|^2 + x_n^2+$(higher-order terms). The Hessian is strictly positive near and at the origin, so it is a convex function of $(z_1, \ldots, z_{n-1}, x_n) \Rightarrow M_+$ is convex $\Rightarrow M_+$ is pseudoconvex.

(3) **M is Levi-pseudoconvex, but not stongly pseudoconvex.**
1st idea: choose holomorphic coordinates such that M_+ is convex - this is not always possible (proof by Kohn-Nirenberg in 1973). We still need to show that $\mathcal{L}_pM \geq 0, \forall p \in M$ near $0 \Rightarrow M_+$ is pseudoconvex near 0.
Idea: Use M^ϵ defined by

$$y_n = \sum_{1\le j,k\le n-1} b_{jk}z_j\overline{z_k} + x_n^2 + \text{(higher-order terms)} + \epsilon(|z_1|^2 + \cdots + |z_{n-1}|^2)$$

$\mathcal{L}_0^{M^\epsilon} = \mathcal{L}_0^M + \epsilon I > 0.$

Exercise 0.156. $\exists \eta > 0$ independent of $\epsilon > 0$ such that $\mathcal{L}_q^{M^\epsilon} > 0$ for $q \in M^\epsilon \cap B(0,\eta), 0 < \epsilon < \epsilon_0$.

$$\begin{aligned} M^\epsilon \text{ strongly pseudoconvex in } B(0,\eta) &\Rightarrow M_+^\epsilon \text{ is pseudoconvex} \\ &\Rightarrow -\log dist(z, M^\epsilon) \text{ is plurisubharmonic in } B(0,\frac{\eta}{2}) \\ &\Rightarrow -\log dist(z, M) \text{ is plurisubharmonic in } B(0,\frac{\eta}{2}) \\ &\Rightarrow M_+ \cap B(0,\frac{\eta}{2}) \text{ is pseudoconvex.} \end{aligned}$$

□

Addenda:

(1) M is strongly pseudoconvex $\Leftrightarrow$ M is locally biholomorphic to a strongly convex hypersurface.
(2) Cannot always do this globally (ex. M = boundary of a slightly thickened, compact totally real submanifold.
(3) M is Levi pseudoconvex $\Leftrightarrow$ M is the limit of a strongly pseudoconvex hypersurface.

An example for last clase: curvature of $y = f(x)$ in $\mathbb{R}^2$ is given by $\frac{y''}{(1+(y')^2)^{\frac{3}{2}}}$.

Homework 2a: $|\frac{z^i w-1}{z^i w+1}| < 1$.

0.1. **Digression on a Related Topic.** Consider $\Omega_1, \Omega_2 \subset \mathbb{C}^n, \phi : \Omega_1 \longrightarrow \Omega_2$ is biholomorphic.
Fact: ϕ maps $\mathbb{C}$-lines to $\mathbb{C}$-lines $\Leftrightarrow$ ϕ is a linear fractional translation (LFT) $z \mapsto \frac{A\cdot\overrightarrow{z}+\overrightarrow{b}}{\overrightarrow{c}^{tr}\cdot\overrightarrow{z}+d}$, $det\begin{pmatrix} A & \overrightarrow{b} \\ \overrightarrow{c}^{tr} & d\end{pmatrix} \neq 0$, A is an $n \times n$ matrix $\Leftrightarrow$ ϕ extends to an automorphism of complex projective space.

Remark 0.157 (Homework 4: #3). ϕ is LFT $\Rightarrow$ $v^{tr}(\frac{\partial^2\phi}{\partial z_j \partial z_k} v$ and $\theta'(z)\cdot v$ is $\mathbb{C}$-linearly dependent.

Recall: $\Omega_1 \xrightarrow{\phi} \Omega_2 \xrightarrow{u} \mathbb{R}$
Recall: $v^{tr}(\text{Hessian of } u\circ\phi)v = v^{tr}\phi'(p)^{tr}(\text{Hessian of } u \text{ at } p)\phi'(p)v + u'(\phi(p))\cdot(v^{tr}(\text{Hessian of } \phi \text{ at } p)v)$
The 2nd term will disappear if v is in the maximal $\mathbb{C}$-subspace of $u \circ \phi = c$. Conclude that $\beta_p = \mathcal{F}_p|_{\text{maximal}\mathbb{C}\text{-subspace}}$ is LFT invariant. We will discuss this further next lecture.

Lecture 29. March 23, 2009

A level set is a set of the form: $\{(z_1, \ldots, z_n) \mid f(z_1, \ldots, z_n) = c\}$, where c is a constant.
Linear Fractional Transformations (LFT):

$$z \xrightarrow{\phi} (\frac{A_{10} + A_{11}z_1 + \cdots + A_{1n}z_n}{A_{00} + A_{01}z_1 + \cdots + A_{0n}z_n}, \ldots, \frac{A_{n0} + A_{n1}z_1 + \cdots + A_{nn}z_n}{A_{00} + A_{01}z_1 + \cdots + A_{0n}z_n})$$

To keep this from degenerating we need $det(A_{jk}) \neq 0$.
Assume that these level sets are submanifolds:

$$A \xrightarrow[LFT]{\phi} B(u = c) \xrightarrow{u} \mathbb{R}, \quad u \circ \phi = c$$

v tangent to (the level set) $u\circ\phi = c$ at $p \in A \Leftrightarrow (u\circ\phi)'(p)\cdot v = u'(\phi(p))\cdot\phi'(p)\cdot v = 0 \Leftrightarrow \phi'(p)\cdot v$ is tangent to $u = c$ at p

Claim: $B_p = \mathcal{F}_p|_{T_pM\cap JT_pM}$ is LFT-invariant. ($\mathcal{L}_p$ = Hermitian part of B_p). Note that this is sometimes referred to as the Bolt form.

Proof. We need $u'(\phi(p))\cdot(v^{tr}(\text{holo. Hessian of } \phi \text{ at } p)v)$. By homework 4, we know that the holomorphic Hessian of phi at p is a $\mathbb{C}$ scalar multiple of $\phi'(p)\cdot v$. We are ok if $\phi'(p)\cdot v$ is in the maximal $\mathbb{C}$-subspace which occurs if and only if v is in the maximal $\mathbb{C}$-subspace. □

Given $\Omega \subset \mathbb{C}^n$ a connected, open set with smooth boundary. Facts:
(1) All $B_p(b\Omega) \geq 0 \Leftrightarrow \Omega\mathbb{C}$-linear convexity $\Leftrightarrow \Omega$ is $\mathbb{C}$-convex
(Ω is $\mathbb{C}$-linearly convex $\Leftrightarrow^{Def.}$ $\mathbb{C}^n \setminus \Omega$ is a union on $\mathbb{C}$-lines.
Ω is $\mathbb{C}$-convex $\Leftrightarrow$ intersections of Ω with $\mathbb{C}$-lines are connected and simply connected.)
(2) $B_p(b\Omega) > 0 \Leftrightarrow^{Def.}$ $b\Omega$ is strongly $\mathbb{C}$-linearly convex $\Leftrightarrow$ $b\Omega$ is locally LFT-equivalent to a strongly convex domain.
(3) Ω above is $\mathbb{C}$-linearly convex $\Leftrightarrow$ $b\Omega$ is a limit of strongly $\mathbb{C}$-linearly convex

Theorem 0.158 (Martineau; 1967)**.** *$\Omega \subset \mathbb{C}^n$ is $\mathbb{C}$-convex, then*
† *$Lu = f$ is solvable on Ω for $L = \sum_{|\alpha|\leq k} c_\alpha \left(\frac{\partial}{\partial z}\right)^\alpha$*
where f is holomorphic on Ω (solution u holomorphic on Ω)

Theorem 0.159. *If Ω is a bounded, pseudoconvex set which satisfies †, then Ω is $\mathbb{C}$-convex.*

ODEs: Fundamental Theorem of ODEs - basic existence, uniqueness results
PDEs:

(1) Cauchy-Kowalevski Theorem - "analytic PDEs are locally solvable" (analytic PDE - all of the data is analytic). The assumption that the PDEs be analytic can be a deal-breaker for some mathematicians - i.e. this assumption is too strong for some.
(2) Constant coefficient linear partial differential operators (LPDO)
(3) Finding solutions can fail with variable coefficients

There is currently no concensus on a fundamental theorem of PDEs. The subject of several complex variables is lurking behind all of these PDE results.

Lecture 30. March 25, 2009

Let $\Omega \subset \mathbb{C}^n$ be open.

Ω is pseudoconvex $\Leftrightarrow \Omega$ admits a (continuous) plurisubharmonic exhaustion function

Theorem 0.160. *Ω is pseudoconvex $\Leftrightarrow \Omega$ admits a C^∞ plurisubharmonic exhaustion function.*

Proof. ($\Leftarrow$) trivial
($\Rightarrow$) Main tool to use is convolutions (details later) □

Theorem 0.161 (Sard's)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and $\Omega^n \xrightarrow{f} \mathbb{R}^k$ be $C^{\max\{n-k+1,1\}}$ smooth. Let $C = \{x \in \Omega \mid rank(f'(x)) < k\}$. Then $f(C)$ has measure zero.*

Definition 0.162. Let $\Omega \xrightarrow{f} \mathbb{R}^n$. Then x_0 is a **critical point** of f if $f'(x) = 0$, in which case $f(x)$ is a **critical value**.

Let $\Omega \subset \mathbb{R}^n$ and $\Omega \xrightarrow{u} \mathbb{R}$ smooth.

Definition 0.163. A point x_0 is a **non-degenerate critical point** if $u'(x_0) = 0$ and the Hessian of u at x_0 is non-degenerate.

Equivalently, if $\Omega \xrightarrow{u'} (\mathbb{R}^n)^*$ sending $x \mapsto u'(x)$ is non-critical at x_0 and $u'(x_0) = 0$.
If x_0 is a non-degenerate critical point then:

- u' is a local diffeomorphism at x_0 (by the inverse function theorem)
- x_0 is an isolated critical point

Theorem 0.164. *If $\Omega \xrightarrow{u} \mathbb{R}$ is smooth, then for almost every $l \in (\mathbb{R}^n)^*$ the function $\tilde{u}(x) = u(x) - l \cdot x$ has no degenerate critical points.*

Proof. $\tilde{u}' = u' - l$:
$\{$bad $l's\} = \{$critical values of $u'\}$ has measure zero by Sard's theorem. □

Corollary 0.165. *If Ω is pseudoconvex, then Ω admits a C^∞ strongly plurisubharmonic exhaustion function without degenerate critical points.*

Proof. If Ω is unbounded, start with an exhaustion function with at leat quadratic growth. □

0.2. **Morse Theory.** Let $u(x_1, x_2, x_3) = x_3$ restricted to the torus on $\mathbb{R}^3$. Imagine the torus standing on one side so that it looks like an O. We are only considering the positive and negative terms in the signature of the Hessian. At the highest point, the Hessian signature is (0,2). At the lowest point the Hessian signature is (2,0). And at the two inside edge points between the high and low the Hessian signature is (1,1). These four points are the critical points of u on the torus. In descending order (accorrding to height) let a, b, c, d be the critical points. The flow is along the gradient of u.
The basins are all the points that flow into a given critical point.

- For the highest point, a, the basin is only that point.
- The point b has a 1-disk as its basin (starting from close to the point a and making a circle between that point and b)
- The point c has a 1-disk as its basin (starting from close to the point b and making a circle between that point and c)
- The point d has a 2-disk as its basin

Differential geometers refer to the disks as cells.

Definition 0.166. Let M be a manifold and $M \xrightarrow{u} \mathbb{R}$ be a smooth exhaustion function with no degenerate critical points. Then u is a **Morse function**.

Theorem 0.167. *Let M be a manifold and $M \xrightarrow{u} \mathbb{R}$ be a Morse function. Then there exists a cell complex $E \subset M$ with one j-dimensional cell for each critical point x_0 with n_- (Hessian of u at x_0)$= j$ such that E is a deformation retract of M. Then $H_j(M, \mathbb{Z}) = 0$ if $j > \max\{n_-(\text{Hessian of } u \text{ at } x_0) \mid x_0 \text{ is a critical point}\}$.*

For references, see:

- Milnor, Morse Theory
- Milnor, k-cobordism theory
- Nicolaescu, Invitation to Morse Theory

Now relating Morse theory to Complex Analysis:
Let u be a strictly plurisubharmonic and Morse, then (by the second lecture) $n_- \leq n$.

Corollary 0.168. *Let $\Omega \subset \mathbb{C}^n$ be pseudoconvex, then $H_j(\Omega, \mathbb{Z}) = 0, \forall j > n$.*

Recall:

$$\begin{aligned} M \text{ is a } \mathbb{C}\text{-submanifold } &\Leftrightarrow T_pM = JT_pM, \forall p \in M \Leftrightarrow M \\ &\Leftrightarrow M \text{ is locally a holomorphic graph } \Omega \xrightarrow{\text{holo.}} \mathbb{C}^{n-k}, \Omega \subset \mathbb{C}^k \end{aligned}$$

Theorem 0.169. *Let M be a closed $\mathbb{C}$-submanifold of $\mathbb{C}^n$, then almost every $z_0 \in \mathbb{C}^n$, $M \xrightarrow{f} \mathbb{R}$ sending $z \mapsto ||z - z_0||^2$ is a strongly plurisubharmonic Morse function.*

Corollary 0.170. *M be a closed $\mathbb{C}$-submanifold of $\mathbb{C}^n$ and $\dim_{\mathbb{C}} M = k$, then M is homotopy equivalent to a k-dimensional cell complex.*

Lecture 31. March 30, 2009

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$

Definition 0.171. A **complex manifold** (or **$\mathbb{C}$-maniifold**) is a Hausdorff topological space X equipped with charts given by homeomorphisms $\{\rho_j : U_j \longrightarrow \Omega_j\}$, where $U_j \subset X$ is open and $\Omega_j \subset \mathbb{C}^n$ is open, such that $\cup U_j = X$ and for any j, k, $\rho_k \circ \rho_j^{-1}$ is holomorphic where it is defined. In addition, each connected component must have a countable dense subset.

Definition 0.172. Let X be a complex manifold with charts given as above. Then f is **holomorphic/plurisubharmonic/etc.** if each $f \circ \rho_j^{-1}$ is holomorphic/plurisubharmonic/etc. where it is defined.

Definition 0.173. Let X and $\widetilde{X}$ be complex manifolds with charts given by $\{\rho_j : U_j \longrightarrow \Omega_j\}$ and $\{\widetilde{\rho_j} : \widetilde{U_j} \longrightarrow \widetilde{\Omega_j}\}$, respectively. $X \xrightarrow{f} \widetilde{X}$ is **holomorphic** if each $\widetilde{\rho_k} \circ f \circ \rho_j^{-1}$ is holomorphic where it is defined.

Definition 0.174. A complex manifold X is **toric** if each Ω_j is a product of $\mathbb{C}$'s and $\mathbb{C}^*$'s and there are only finitely many Ω_j's.

$$\rho_k \circ \rho_j^{-1} : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n \quad \text{given by} \quad z \mapsto (z^{\alpha_1}, \ldots, z^{\alpha_n}) \quad M_{jk} = (\alpha_1 \alpha_n)^{tr}$$

inv. integer matrix with determinant not equal to 1.

Theorem 0.175. *Let X be toric manifold with top row entries of M_{1k} positive for all k. Let $K \subset X$ be compact and $X \setminus K$ connected. If f is holomorphic on $X \setminus K$, then f extends to a holomorphic function on X.*

Remark 0.176. This theorem was presented as the AMS meeting on March 27, 2009.

We want to look at complex manifolds that ressemble pseudoconvex domains.

Definition 0.177. Let X be a $\mathbb{C}$-manifold of $\mathbb{C}$-dimension n. X is **Stein** if X satisfies:

(1) holomorphic convexity
(2) $\forall p \in X, \exists f_1, \ldots, f_n$, where $X \xrightarrow{f_j} \mathbb{C}$ is holomorphic, and such that $f_1'(p), \ldots, f_n'(p)$ are $\mathbb{C}$-linearly independent.

Remark 0.178. If $X \subseteq \mathbb{C}^n$ is a $\mathbb{C}$-manifold of dimension n, then the second condition is satisfied by setting $f_j(z) = z_j$.

Theorem 0.179. *A $\mathbb{C}$-manifold is Stein $\Leftrightarrow$ X has a strongly plurisubharmonic exhaustion function.*

Proof. ($\Rightarrow$) Like earlier in the course and ($\Leftarrow$) Like the Levi problem. □

Theorem 0.180 (Bishop/Narasimhan, 1961)**.** *A $\mathbb{C}$-manifold is Stein $\Leftrightarrow$ X is biholomorphic to a closed $\mathbb{C}$-manifold of $\mathbb{C}^N$ (for some $N \geq n$).*

Proof. ($\Rightarrow$) Like Whitney Embedding theorem and ($\Leftarrow$) Trivial. □

What values of N work in the previous theorem?
Bishop/Narasimhan: $N = 2\dim_{\mathbb{C}}(X) + 1$
Eliashberg-Gromov: $N = \frac{3}{2}(\dim_{\mathbb{C}}(X) + 1)$
Open Question: $N = 2$ for $\dim_{\mathbb{C}}(X) = 1$
– *Globevnik-Stensones:* OK for $X \subset \mathbb{C}$ with smooth boundary.

Let X be a Stein manifold with $\dim_{\mathbb{C}}(X) = k$. Then X admits a Morse function with $n_- \leq k$ at each critical point. Also (by Morse Theory) X is homotopy equivalent to a cell complex of real dimension at most k.

Theorem 0.181 (Eliasberg, 1990's)**.** *If M is an $\mathbb{R}$-manifold with $\dim_{\mathbb{R}} M = 2k, k > 2$, then M is diffeomorphic to a Stein manifold $\Leftrightarrow$ M admits a smoothly varying $J_p : T_pM \longrightarrow T_pM$ such that $J^2 = -I$ (i.e. M is "almost a $\mathbb{C}$-manifold) and M admits a Morse function with $n_- \leq k$ at each critical point.*

Remark 0.182. There is a more complicated result for $k = 2$. We probably need to assume that M is atleast C^2 so that we can consider the Hessian.

Open Question: When is M diffeomorphic to a pseudoconvex in $\mathbb{C}^k$?

Lecture 32. April 1, 2009

Assume that everything is smooth (although we could make a less stringent assumption)
A vector field L on a manifold M assigns to a point $x \in M$ a tangent vector $L(x) \in T_xM$.
Get a differential operator: for a smooth map $M \xrightarrow{f} \mathbb{R}$, $(Lf)(x) = f'(x) \cdot L(x)$. Let $M \subset \mathbb{R}^n$ be open. Identify T_xM with $\mathbb{R}^n$. $M \xrightarrow{L} \mathbb{R}^n$ by $x \mapsto (a_1(x), \dots, a_n(x))$ where $Lf = (a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n})f$. Let $\tilde{L} = b_1 \frac{\partial}{\partial x_1} + \cdots + b_n \frac{\partial}{\partial x_n}$. Then Lie bracket is:

$$[L, \tilde{L}] = L\tilde{L} - \tilde{L}L = \sum_{j,k} (a_k \frac{\partial b_j}{\partial x_k} - b_k \frac{\partial a_j}{\partial x_k}) \frac{\partial}{\partial x_j})$$

This is also a vector field.

Exercise 0.183. $[\alpha L, \beta \tilde{L}] = \alpha\beta[L, \tilde{L}] + \alpha(L\beta)\tilde{L} - \beta(\tilde{L}\alpha)L$

We want to study: † $\quad L_1 f = \cdots = L_s f = 0$
† $\quad \Rightarrow (\varphi_1 L_1 + \dots \varphi_s L_s)f = 0$
$span\{L_1, \dots, L_s\} \equiv \{\varphi_1 L_1 + \cdots + \varphi_s L_s \mid M \xrightarrow{\varphi_j} \mathbb{R}\}$
View $L_1, \dots, L_s$ as equivalent to $\tilde{L_1}, \dots, \tilde{L_s}$ if they have the same span.
Special Case: $L_j = \frac{\partial}{\partial x_j}, 1 \leq j \leq s < n$
f satisfies † $\Leftrightarrow$ f is (locally) a function of $x_{s+1}, \dots, x_n$.
The span of $\{L_1, \dots, L_s\}$ is closed under the Lie bracket.

Theorem 0.184 (Frobenius)**.** *If $span\{L_1, \dots, L_s\}$ is closed under the Lie bracket and $L_1(x), \dots, L_s(x)$ are linearly independent at x, then there exists a change of coordinates near x converting the span of $\{L_1, \dots, L_s\}$ to the span of $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s}\}$.*

When $s = 1$ there is a slightly stronger result: if $L \neq 0$ at x, then L is locally equivalent to $\frac{\partial}{\partial x_1}$. This is the *"Fundamental Theorem of ODEs"* - reference: Arnold "ODEs."

Proof. See Narasimhan's result Analysis on Manifolds or Sharpe's Differential Geometry □

Remark 0.185. There is also a holomorphic version of this result (i.e. a version for holomorphic vector fields).

"Almost $\mathbb{C}$-structure" on a manifold M is $J_x : T_xM \longrightarrow T_xM$, $\mathbb{R}$-linear such that $J^2 = -I$ (need $\dim_{\mathbb{R}} M$ to be even).
The subscript *st* stands for "standard." J_{st} on $\mathbb{R}^{2n}_{(x_1, y_1, \dots, x_n, y_n)}$:

$$J_{st}(\frac{\partial}{\partial x_k}) = \frac{\partial}{\partial y_k} \text{ and } J_{st}(\frac{\partial}{\partial y_k}) = -\frac{\partial}{\partial x_k}$$

Suppose that J can be converted by a local change of coordinates to J_{st}. Then these changes of coordinates give holomorphic charts and M becomes an "honest" $\mathbb{C}$-manifold.
Given J and a vector field X, then $X + iJX$ is a type (0,1) vector field and $X - iJX$ is a type (1,0) vector field.
For J_{st}:
Type (0,1) vector fields have the form $\sum a_j(z) \frac{\partial}{\partial \bar{z}_j}$
L is type (0,1) $\Leftrightarrow Lh \equiv 0, \forall h$ holomorphic
L_1, L_2 are type (0,1) $\Rightarrow [L_1, L_2]h = L_1 L_2 h - L_2 L_1 h \equiv 0, \forall h$ holomorphic $\Rightarrow [L_1, L_1]$ is type (0,1)
So the set of (0,1) vector fields is closed under the Lie bracket. (‡)
Condition ‡ is necessary for J to be locally equivalent to J_{st}. Let X, Y be real vector fields.

$$[X + iJX, Y + iJY] = [JX, Y] + [X, JY] + i([JX, Y] + [X, JY])$$

Need $J([X,Y]-[JX,JY])=[JX,Y]+[X,JY]$

Exercise 0.186. $J([X,Y]-[JX,JY])=[JX,Y]+[X,JY]$ is automatic in real dimension 2.

Theorem 0.187 (Newlander-Nirenberg)**.** *J is locally equivalent to J_{st} $\Leftrightarrow$ J satisfies $J([X,Y]-[JX,JY])=[JX,Y]+[X,JY]$.*

Definition 0.188. J is **integrable** if J is locally equivalent to J_{st}.

Exercise 0.189. $f=(f_1,\dots,f_n)$: (a piece of)$M \longrightarrow \mathbb{C}^n$ converts J to J_{st} $\Leftrightarrow$ $Lf_j=0$ for L a (0,1) vector-field (with respect to J) - "f_j is J-holomorphic" - and $f_1',\dots,f_n'$ are linearly indepenedent.

In $\mathbb{C}$-dimension 1:
C^ω (real-analytic): Gauss
C^∞: Beltrami
"L^∞": Ahlfors-Bers
In $\mathbb{C}$-dimension > 1:
C^ω (real-analytic): easy proof
C^∞: Neqlander-Nirenberg
"L^∞": Not yet avaialble

C^ω case. Let $X=\sum a_j(x)\frac{\partial}{\partial x_j}$ be a C^ω vector field near $0\in\mathbb{R}^{2n}$ and $\tilde{X}=\sum a_j(z)\frac{\partial}{\partial z_j}$ a holomorphic vector field near $0\in\mathbb{C}^{2n}$.

$$J:(\text{neighborhood of } 0\in\mathbb{R}^{2n}) \longrightarrow \{\text{real } 2n\times 2n \text{ matrices } J \mid J^2=-I\}$$
$$\tilde{J}:(\text{neighborhood of } 0\in\mathbb{C}^{2n}) \longrightarrow \{\text{complex } 2n\times 2n \text{ matrices } J \mid J^2=-I\}$$

$\{X+iJX\}$ is closed under the Lie bracket $\Rightarrow$ $\{\tilde{X}+i\tilde{J}\tilde{X}\}$ is closed under the Lie bracket. By the Holomorphic Frobenius theorem, $\exists \tilde{f}_1,\dots,\tilde{f}_n$ holomorphic near $0\in\mathbb{C}^{2n}$ such that $(\tilde{X}+i\tilde{J}\tilde{X})\tilde{f}_j=0$ and $\tilde{f}_1,\dots,\tilde{f}_n$ are linearly independent. Restrict $\tilde{f}$ to $\mathbb{R}^{2n}$ to get J-holomorphic f_j with $f_1',\dots,f_n'$ linearly independent. ☐

Lecture 33. April 3, 2009

Facts

- S^n admits an almost $\mathbb{C}$-structure $\Leftrightarrow$ $n=2,6$
- S^n admits an "honest" $\mathbb{C}$-structure $\Leftrightarrow$ $n=2$

For a typical non-integrable J:

- All J-holomorphic maps f : (open subset of M) $\longrightarrow \mathbb{C}$ are constant
- All J-holomorphic maps f : (open subset of $\mathbb{C}^n$) $\longrightarrow M$ have rank $f'\le 1$

There are many J-holomorphic maps $f:\Delta\longrightarrow M$ *(this is very important in symplectic geometry).*

Let M be a real hypersurface with defining function ρ and let X be a vector field on a neighborhood of M.

$$X \text{ is tangent to } M \Leftrightarrow X\rho\equiv 0 \text{ on } M \Leftrightarrow Xf\equiv 0 \text{ on } M, \forall f=0 \text{ on } M$$
$$X,Y \text{ tangent to } M \Rightarrow [X,Y]f=XYf-YXf=0 \text{ on } M, \forall f=0 \text{ on } M \Rightarrow [X,Y] \text{ tangent to } M$$

This is valid for M of any dimension.

Now suppose that M is a CR-submanifold of $\mathbb{C}^n$, $H_pM=T_pM\cap JT_pM$ has constant dimension.

Proposition 0.190. $\mathcal{L}_p : H_pM \times H_pM \longrightarrow T_p\mathbb{C}^n/T_pM$. *If* $X(q), Y(q) \in H_qM, \forall q \in M$, *then* $\mathcal{L}_p(X(p), Y(p)) = -J[X, JY](p)$ *is a well-defined symmetric Hermitian form.*

Proof.

$$\mathcal{L}_p(JX(p), JY(p)) = -J[JX, -Y](p) = -J[Y, JX] = \mathcal{L}_p(Y(p), X(p))$$

$$\mathcal{L}_p(Y(p), X(p)) = J[JX, Y](p) = (-J[X, JY]-[X, Y]+[JX, JY])(p) = -J[X, JY](p) = \mathcal{L}_p(X(p), Y(p))$$

If X, Y are tangent to M, then JX, JY are tangent to M and the bracket of any combination of these is also tangent to M. We still need to show that $\mathcal{L}_p(X(p), Y(p))$ depends only on $X(p), Y(p)$ (i.e. if $X(p) = \tilde{X}(p)$, then $\mathcal{L}_p(X(p), Y(p)) = \mathcal{L}_p(\tilde{X}(p), Y(p))$). Equivalently, $X(p) = 0 \Rightarrow \mathcal{L}_p(X(p), Y(p)) = 0$ and $X = \sum f_j X_j, f_j(p) = 0$. It suffices to show that $\mathcal{L}_p(fX(p), Y(p)) = f(p)\mathcal{L}_p(X(p), Y(p))$.

$$\begin{aligned}\mathcal{L}_p(fX, Y) &= J[fX, JY] \\ &= -J(f[X, JY] + (JYf)X) \\ &= -fJ[X, JY] - (JYf)JX, \quad JYf \text{ is tangent to } M \\ & f(p)\mathcal{L}_p(X(p), Y(p))\end{aligned}$$

□

Remarks:

(1) If M is a real hypersurface, then this definition agrees with the definition given earlier.

Proof. First reduce to $y_n = |z_1^2| + \cdots + |z_{n-1}^2| +$ (higher-order terms). Then use the vector field: $\frac{\partial}{\partial x_j} - 2x_j \frac{\partial}{\partial y_n} +$ (higher-order terms). □

(2) Suppose that S is a complex-submanifold of M. The vector fieldds tangent to S are closed under $[,] \Rightarrow \mathcal{L}_p(X(p), Y(p)) = 0$ for $X(p), Y(p) \in T_pS$. If M is a strongly pseudoconvex hypersurface, then M contains no (non-trivial) $\mathbb{C}$-submanifolds.

(3) $\mathcal{L} = 0$ on $M \Leftrightarrow$ {vector fields with values in H_pM are closed under the Lie bracket.}. By the Frobenius theorem, this implies that M decomposes locally into disjoint complex submanifolds S_t with $T_pS_t = H_pM$. This is a foliation by $\mathbb{C}$-submanifolds.

Lecture 34. April 6, 2009

Definition 0.191. An **"abstract" CR-manifold** is a manifold M equipped with a subbundle $HM \subset TM$ and linear maps $J_p : H_pM \longrightarrow H_pM$ such that:

(1) $J^2 = -I$

(2) If X, Y are HM-valued vector-fields, then:

$$[X, Y] - [JX, JY] \text{ is } HM\text{-valued and } J([X, Y] - [JX, JY]) = [JX, Y] + [X, JY]$$

The Levi-form:

$$H_pM \times H_pM \xrightarrow{\widetilde{\mathcal{L}_p}} T_pM/H_pM \text{ given by } (X, Y) \mapsto [X, JY], \text{ where } X, Y \text{ are } HM\text{-valued}$$

- If everything is C^ω (i.e. real analytic), then M is locally equivalent to a CR-submanifold of $\mathbb{C}^N$.
- If $\dim_{\mathbb{R}} M = \dim_{\mathbb{R}} HM + 1 \geq 7, \widetilde{\mathcal{L}} > 0, C^\infty$ then M is locally equivalent to a CR-submanifold of $\mathbb{C}^N$.
- If $\dim_{\mathbb{R}} M = \dim_{\mathbb{R}} HM + 1 = 3, \widetilde{\mathcal{L}} > 0, C^\infty$, then there is no CR-embedding.

0.3. **Zero-sets of Holomorphic Functions:** If $n = 1$, if f is a holomorphic function that is not identically zero, then near z_0, f can be factored as: $f(z) = (z - z_0)^n h(z)$, where h is holomorphic and $h(z_0) \neq 0$. In this case, "f has a zero of order n at z_0."
Now consider $n > 1$. Let f be holomorphic on a convex neighborhood of $0 \in \mathbb{C}^n$, $f(0) = 0$, and f is not identically zero. Look at f along $\mathbb{C}$-lines through 0:
Given a $\mathbb{C}$-line L through 0, f has a zero of order: $ord_L(f)$ at 0 along L. Let $m = ord(f) = \min_L(ord_L(f))$.

Example 0.192. Let $f(z_1, z_2) = z_1^2 - z_2^3$. Then $ord_{z_2-\text{axis}}(f) = 3$ and $ord_L(f) = 2$ for all other lines through 0. Therefore $ord(f) = 2$.

After a linear change of coordinates we may assume that $m = ord(f) = ord_{z_n-\text{axis}}(f)$ (i.e. $f(0, \ldots, 0, z_n)$ has a zero of order m at $z_n = 0$).
Special Case: If $m = 1$ and $\frac{\partial f}{\partial x_n} \neq 0$, then by the implicit function theorem $\{f = 0\}$ is local graph of a holomorphic function and $\{f = 0\}$ is a $\mathbb{C}$-submanifold of $\mathbb{C}^n$. So we know pretty much what happens when $m = 1$.
General Case: Assume $m > 1$. For a fixed $z' = (z_1, \ldots, z_{n-1})$ near 0, let $h_{z'}(z_n) = f(z', z_n)$. Pick $\epsilon > 0$ such that h_0 has m zeros in $|z_n| \leq \epsilon$. By the argument principle,

$$\#\text{of zeros of } h_{z'} \text{ inside } |z_n| = \epsilon \text{ is } \frac{1}{2\pi i}\int_{|\zeta|=\epsilon} \frac{\frac{\partial f}{\partial z_n}(z', \zeta)}{f(z', \zeta)} d\zeta$$

Holomorphic function of z' near 0 is integer valued. Therefore $h_{z'}$ has m zeros inside $|z_n| = \epsilon$ when z' is near 0. Let these m zeros be given by: $\alpha_1(z'), \ldots, \alpha_m(z')$.
More generally, if $a(z)$ is holomorphic near 0, then

$$\frac{1}{2\pi i}\int_{|\zeta|=\epsilon} \frac{a(z', \zeta)\frac{\partial f}{\partial z_n}(z', \zeta)}{f(z', \zeta)} d\zeta = \sum_{j=1}^{m} a(z_j'\alpha_j(z'))$$

This is holomorphic for z' near 0.

Suppose $g(z)$ is holomorphic near 0. Let $a(z) = \log(1 - \delta g(z))$, which is holomorphic near $z = 0$ for δ small. Conclude:

$$\sum_{j=1}^{m} \log(1 - \delta g(z', \alpha_j(z')) \text{ is holomorphic near } z' = 0 \text{ for small } \delta$$

Exponentiating, we see that:

$$\prod_{j=1}^{m}(1 - \delta g(z', \alpha_j(z')) = \sum_{j=0}^{m} c_j(z')\delta^j \text{ is holomorphic near } z' = 0,$$

$$\text{where } c_j(z') = \frac{1}{2\pi i}\int_{\zeta=\epsilon} \prod_{j=1}^{m}(1 - \delta g(z', \alpha_j(z'))\frac{\partial \zeta}{\zeta^{j+1}}, \text{ which is holomorphic near } z' = 0$$

$$\text{In particular, } c_0(z') = 1$$

$$c_1(z') = -\sum_{j=1}^{m} g(z', \alpha_j(z'))$$

$$c_2(z') = \sum_{j<k} g(z', \alpha_j(z'))g(z', \alpha_k(z')), \ldots,$$

$$c_m(z') = (-1)^m \prod_{j=1}^{m} g(z', \alpha_j(z'))$$

Definition 0.193. The **Weierstrass polynomial**, $W(z)$, for f at 0 is:

$$W(z) = \prod_{j=1}^{m}(z_n - \alpha_j(z')) = z_n^m + b_1(z')z_n^{m-1} + \cdots + b_n(z'),$$

where all $b_j(z')$ are holomorphic near $z' = 0$, "$b_j = c_j$ with $g(z) = z_n$."

Lecture 35. April 8, 2009

Some algebra:

$$(z-w_1)\dots(z-w_n) = z^n - e_1(w_1,\dots,w_n)z^{n-1} + e_2(w_1,\dots,w_n)z^{n-2} - \dots (-1)^n e_n(w_1,\dots,w_n),$$

$$\text{where } e_1(w_1,\dots,w_n) = \sum_j w_j, \dots, e_n(w_1,\dots,w_n) = \prod_j w_j$$

$$\prod_{j<k}(w_j - w_k)^2 = (-1)^{\frac{n(n-1)}{2}} detA, \quad \text{where } A \text{ is a matrix involving the } e_j's$$

Example 0.194. $\{z_1 - z_2^2 = 0\} = \{z_1 = z_2^2\} = \{z_2 = \sqrt{z_1}\}$ is a 1-dimensional $\mathbb{C}$-manifold.

Example 0.195. $\{z_1^2 - z_2^2 = 0\} = \{z_2 = \pm z_1\}$ is a union of two 1-dimensional $\mathbb{C}$-manifolds.

Example 0.196. $\{z_1^3 - z_2^2 = 0\} = \{z_2 = z_1^{\frac{3}{2}}\}$ is not a finite union of 1-dimensional $\mathbb{C}$-manifolds.

Given a function f which is holomorphic near 0 and $f \neq 0$, after a change in coordinates:

$$f(0,\dots,0,z_n) \text{ has } m \text{ zeros in } |z_n| \leq \epsilon \text{ all at } z_n = 0$$

For $z' = (z_1,\dots,z_{n-1})$ small, $f(z', z_n)$ has m zeros given by $\alpha_1(z'),\dots,\alpha_n(z')$ in $|z_n| \leq \epsilon$ (none on $|z_n| = \epsilon$),
Recall: if g is holomorphic near 0, then the elementary symmetric functions of $g(z', \alpha_j(z'))$ are holomorphic near $z' = 0$.

$$\text{Let} \quad h(z) = \frac{1}{2\pi i}\int_{|\zeta|=\epsilon} \frac{f(z',\zeta)}{w(z',\zeta)}\frac{d\zeta}{\zeta - z_n}$$

This is holomorphic near $z = 0$ and $\frac{f(z',\zeta)}{w(z',\zeta)}$ has no poles. Then $h(z) = \frac{f(z',z_n)}{w(z',z_n)}$, where this makes sense. So that $f(z) = W(z)h(z)$, where W is the Weierstrass polynomial and h is holomorphic near 0. Near zero, $f(0,z_n) = z_n^m h(0,z_n)$ and no more zeros can be pulled out, so $h(0) \neq 0$.

Theorem 0.197 (Weierstrass Preparation)**.** *$f(z) = W(z)h(z)$, where h is holomorphic near 0 and $h(0) \neq 0$.*

Corollary 0.198. *Zeros of f are not isolated.*

Corollary 0.199. *Suppose g is holomorphic near 0, f is a function vanishing near the origin (as we assumed previously) and $|g|$ has a local maximum at 0 along $\{f = 0\}$. Then g is constant on $\{f = 0\}$.*

Proof. Since $|g|$ has a local max at 0 along $\{f = 0\}$, $|\sum_{j=1}^m g(z', \alpha_j(z'))|$ is holomorphic near $z' = 0$ and it has a local max at $z' = 0$ (by the triangle inequality). Then $\sum_{j=1}^m g(z', \alpha_j(z'))$ is constant near $z' = 0$. Therefore $g(z', \alpha_j(z')) = g(0)$. □

Corollary 0.200. *Let f be holomorphic on Ω, $f \neq 0$, and let g be holomorphic and bounded on $\Omega \setminus \{f = 0\}$. Then g extends uniquely to a holomorphic function on Ω.*

Proof. Localize, reduce to f as above. For fixed z', $g(z', z_n)$ is a bounded holomorphic function of z_n with isolated (removable) singularities at $\alpha_j(z')$. By Cauchy's theorem:

$$g(z', z_n) = \frac{1}{2\pi i} \int_{|\zeta|=\epsilon} g(z', \zeta) \frac{d\zeta}{\zeta - z_n},$$

which is a holomorphic function at z near 0. □

Theorem 0.201 (Rado's)**.** *Given* $\Omega \xrightarrow{f} \mathbb{C}$ *continuous and* f *holomorphic on* $\Omega \setminus \{f = 0\}$*, then* f *is holomorphic on* Ω*.*

Proof. Using Cauchy integrals as in the previous proof, it suffices to prove this on slices so we can focus on the case when $n = 1$. The problem is local, so assume that $\Omega = \Delta \subset \mathbb{C}$, f continuous on $\overline{\Delta}$ and $|f| < 1$ on $\overline{\Delta}$. Let $E = \{f = 0\}$. Let g be the continuous function of $\overline{\Delta}$ which solves the Dirichlet problem: g harmonic on Δ and $g = f$ on $b\Delta$. Define φ for $\epsilon > 0$ as:

$\varphi = Re(f - g) + \epsilon \log |f|$ this is harmonic on $\Delta \setminus E$ and has " negative boundary values"

By the maximum principle (since φ is negative near the boundary), $\varphi \leq 0$ on $\overline{\Delta}$. Let $\epsilon \to 0$, then $Re(f - g) \leq 0$ on $\overline{\Delta}$. Similarly, $Re(f - g) \geq 0, Im(f - g) \leq 0, Im(f - g) \geq 0 \Rightarrow f = g$. f is holomorphic on $(\Delta \setminus E) \cup Int(E)$ (dense open subset of Δ), so f is holomorphic on Δ. □

Lecture 36. April 10, 2009

Example 0.202. $(w_2 - 2_1)^2 = (w_1 + w_2)^2 - 4w_1w_2 = e_1^2 - 4e_2 = -\det \begin{pmatrix} 1 & -e_1 & e_2 \\ 2 & -e_1 & 0 \\ 0 & 2 & -e_1 \end{pmatrix}$

Theorem 0.203 (Holomorphic Inverse Function Theorem, Version 2)**.** *Given* $\Omega_1, \Omega_2 \subseteq \mathbb{C}^n$ *open and* $\Omega_1 \xrightarrow{f} \Omega_2$ *holomorphic bijection with continuous inverse, then* f^{-1} *is holomorphic.*

Proof. Assume that Ω_1, Ω_2 are connected, bounded and $\det f' = 0$. Then (by Sard's Theorem) So $\det(f')$ is not zero. Let $g(w) = \det(f'(f^{-1}(w))$ on Ω_2. Then g is continuous on Ω_2 and g is holomorphic on $\Omega_2 \setminus \{g = 0\}$ (by the holomorphic inverse function theorem, version 1). By Rado's theorem, g is holomorphic on Ω_2. $(f^{-1})_k$ (i.e. the k-th component of f^{-1}) is a bounded, holomorphic function on $\Omega_2 \setminus \{g = 0\}$. The removable singularity result from last lecture implies that f^{-1} is holomorphic on Ω_2. □

Remark 0.204. The assumption that f^{-1} is continuous is unnecessary, however without this assumption the proof is a bit harder. See Rudin's Function Theory in the Unit Ball.

Weierstrass Preparation Theorem

f holomorphic near 0 but not identically zero, then after a linear change of coordinates:

$$f(z) = \prod_{j=1}^{m} (z_n - \alpha_j(z')) \cdot h(z) = W(z) \cdot h(z), \text{ in } ||z'|| < \epsilon, ||z_n|| < \epsilon$$

$h(z)$ is a non-vanishing, holomorphic function and $W(z)$ is the Weierstrass polynomial and its coefficients are holomorphic for $||z'|| < \epsilon$ with all $\alpha_j(0) = 0$. Let k be the maximum number of distinct roots.

Suppose $m = 5, n = 2$. Then there will be one triple root, which varies holomorphically in z' and one double root which varies holomorphically with z'. In general, this shows that:

$$U \equiv \{z' \mid ||z'|| < \epsilon, W(z', z_n) \text{ has } k \text{ distinct roots}\} \neq \emptyset$$

is an open subset of $||z'|| < \epsilon$. Let

$$\delta(z') = \begin{cases} \text{discriminant of distinct roots} & \text{for } z' \in U \\ 0 & \text{for } z' \notin U \end{cases}$$

δ is continuous on $||z'|| < \epsilon$ and holomorphic on $U = \{||z'|| < \epsilon\} \setminus \{\delta(z') = 0\}$. By Rado's theorem, δ is a holomorphic function on $||z'|| < \epsilon$. $\{f(z) = 0\} \setminus \{f(z) = 0, \delta(z') = 0\}$ is an $(n-1)$-dimensional $\mathbb{C}$-submanifold and $\{f(z) = 0, \delta(z') = 0\}$ is a small "exceptional" set.

Example 0.205. **When** $n = 2$**:** the set of "exceptional" points is (locally) finite and (globally) a countable discrete set. So that $\{f(z) = 0\} \setminus$ (finite set) is a 1-dimensional $\mathbb{C}$-manifold.
When $n > 2$**:** f is holomorphic on $\Omega \subseteq \mathbb{C}^n$ is a connected open set and f is not identically zero, then $\{f = 0\} = X_{n-1} \cup X_{n-2} \cup \cdots \cup X_0$, where X_j is a closed, j-dimensional $\mathbb{C}$-submanifold of $\Omega \setminus (X_0 \cup \cdots \cup X_{j-1})$.

Definition 0.206. An **analytic subset** X of Ω is a closed subset locally described as $\{f_1(z) = \cdots = f_k(z) = 0\}$.

This gives the same stratification result.

Remark 0.207. Some good references are:
Chirka's, Complex Analytic Sets
Fritzche's and Grauert's, From Holomorphic Functions to . . . (chapter III).

Recall: A function $\Omega \xrightarrow{u} \mathbb{R}$ which is C^2 is:

$$\begin{aligned} \text{plurisubharmonic} &\Leftrightarrow \sum_{j,k} u_{j,\overline{k}} a_j \overline{a_k} \geq 0 \text{ on } \Omega, \forall a \in \mathbb{C}^n \\ &\Leftrightarrow \int_\Omega \sum_{j,k} u_{j,\overline{k}} a_j \overline{a_k} \varphi \geq 0, \forall a \in \mathbb{C}^n, \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \\ &\Leftrightarrow \dagger \quad \int_\Omega u\Big(\sum a_j a_{\overline{k}} \varphi_{j,\overline{k}}\Big) \geq 0, a \in \mathbb{C}^n, \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \end{aligned}$$

The a_l is a constant and $u_{j,\overline{k}} l$ is the derivative of u.

Lecture 37. April 13, 2009

Theorem 0.208. *Let* $\Omega \subset \mathbb{C}^n$ *be an open connected set. If* $u \in L^1_{loc.}(\Omega)$ *satisfying* $\dagger$*, then* $\exists! v : \Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$ *such that:*

(1) $u = v$ *almost everywhere*
(2) v *satisfies the sub-averaging property*
(i.e. $v(\alpha(0)) \leq avg_\theta v(\alpha(e^{i\theta})), \forall \mathbb{C}$*-affine maps* $\overline{\Delta} \xrightarrow{\alpha} \Omega$*)*
(3) $v(z) \geq \limsup_{\zeta \to z} v(\zeta)$ *(i.e.* v *is "upper semi-continuous")*
(4) $v \neq -\infty$

$\Omega \xrightarrow{v} \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic $\Leftrightarrow^{Def.}$ v satisfies conditions 2-4 in the previous theorem
$\Rightarrow v$ satisfies the assumptions of the previous theorem

Remark 0.209. A good reference is Hormander's monograph.

Example 0.210. If f is holomorphic on Ω, then $\log|f|$ is plurisubharmonic on Ω.
To show this, it suffices to check that if f is holomorphic on a neighborhood of $\overline{\Delta} \subset \mathbb{C}$, then $\log|f(0)| \leq avg_\theta \log|f(e^{i\theta})|$.

Let $z_1, \dots, z_m$ be the zeros of f in Δ. Then $f(z) = \left(\prod_{j=1}^{m} \frac{z-z_j}{1-\overline{z_j}z}\right) e^{g(z)} \equiv B(z)e^{g(z)}$ on $\overline{\Delta}$. $g(z)$ is holomorphic on $\overline{\Delta}$ and $B(z)$ is a Blashcke product so $|B(z)| = 1$ on $b\Delta$.

$$\log|f(0)| = \log|B(0)| + \log|e^{g(0)}| = 0 + |Re(g(0))| \leq avg_\theta Re(g(e^{i\theta})) = avg_\theta \log|f(e^{i\theta})|$$

Example 0.211. Suppose $f_1, \dots, f_k$ are holomorphic on Ω. Then $\log(|f_1|^2 + \cdots + |f_k|^2)$ is plurisubharmonic on Ω. This follows from the previous example and an example from the lecture on March 29.

Example 0.212. Let $v(z) = \sum_{n>0} \frac{\log|z-\frac{1}{n}|}{n^2}$ is subharmonic on $\mathbb{C}$ (recall: subharmonic is plurisubharmonic when $n = 1$). Then $v(\frac{1}{n}) = -\infty$ and $v(0) > -\infty$, so v is not "continuous in the generalized sense" (i.e. including $-\infty$).

FACTS:

(1) Let $v_1, v_2, \dots$ be plurisubharmonic on Ω and $v_1 \geq v_2 \geq \dots$, then $\lim v_j$ is either plurisubharmonic or $-\infty$.
(2) v is plurisubharmonic $\Leftrightarrow$ locally $\exists$ smooth plurisubharmonic functions satisfying $v_1 \geq v_2 \geq \dots$ and such that $v_j \to v$. *(To show this, define v to be the pointwise limit of the convolutions of v_j.)*

Theorem 0.213. *If Ω is a pseudoconvex domain, then Ω has a C^∞ strongly plurisubharmonic exhaustion function, ρ.*

Proof. Since Ω is pseudoconvex, we have a continuous plurisubharmonic exhaustion function ψ for Ω. Let $\psi_j \in C^\infty(\Omega)$ such that $\{\psi_j$ is strongly plurisubharmonic and $|\psi_j - \psi| < \frac{1}{10}\}$ on $\{\psi \leq j\}$. How can we find such a ψ_j? Let

$$\psi_j(z) = \delta||z||^2 + \left(\text{convolution-approximation of } \psi \cdot \chi_{\{\psi \leq j+1\}}\right)$$

Pick $\eta \in C^\infty(\mathbb{R})$ such that $\eta = 0$ on $(\infty, 0)$ and η is strictly increasing and strictly convex on $(0, \infty)$. Set

$$\rho = \rho_0 + \sum c_j \eta \circ (\psi_j + 2 - j),$$

where $c_j > 0$ and $\eta \circ (\psi_j + 2 - j)$ is plurisubharmonic for $\psi(z) \leq j$. In fact, ρ is strongly plurisubharmonic for $j - 1 \leq \psi(z) \leq j$ and vanishes for $j \geq \psi(z) + 3$. On a neighborhood of $\{k - 1 \leq \psi(z) \leq k\}$ we have $j \leq k - 1$ no control, $j = k$ strongly plurisubharmonic and positive, $j \geq k + 1$ plurisubharmonic, and $j \geq k + 3$ identically zero. Choose $c_k > 0$ large enough so that ρ is strongly plurisubharmonic and $\rho \geq \psi$ on $\{k - 1 < \psi(z) \leq k\}$. ☐

Corollary 0.214. *If Ω is a pseudoconvex domain, then there exist smooth, bounded, strongly pseudoconvex domains $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega$ such that every compact $K \subset \Omega$ is contained in some Ω_j.*

Proof Of Corollary. Let $\Omega_j = \{z \in \Omega \mid \rho(z) < \nu_j\}$, where ρ is the exhaustion function of Ω given by the previous theorem and $\nu_1 < \nu_2 < \cdots \to \infty$ where each v_j is not a critical value. ☐

Lecture 38. April 15, 2009

Definition 0.215. $\Omega \subset \mathbb{C}^n$ open is **circular** if $e^{i\theta}\Omega = \Omega, \forall \theta \in \mathbb{R}$.

Definition 0.216. $\Omega \subset \mathbb{C}^n$ open is **complete circular** if $\lambda\Omega \subset \Omega, \forall \lambda \in \overline{\Delta}$.

If f is holomorphic on a complete circular set Ω and $f \in L^1(\Omega)$, then

$$\int_0^{2\pi} \int_\Omega f(e^{i\theta}z) dv(z) d\theta = \int_\Omega \int_0^{2\pi} f(e^{i\theta}z) d\theta dv(z).$$

$\int_0^{2\pi}\int_\Omega f(e^{i\theta}z)dv(z)d\theta = 2\pi\int_\Omega f dv$ and by the Mean Value Theorem, $\int_\Omega\int_0^{2\pi} f(e^{i\theta}z)d\theta dv(z) = 2\pi f(0)vol(\Omega)$. Therefore

$$f(0) = \frac{1}{vol(\Omega)}\int_\Omega f dV$$

Assume $p \geq 1$. Then by Holder's inequality,

$$|f(0)| \leq \frac{||f||_p||1||_q}{vol(\Omega)} = \frac{||f||_p}{(vol(\Omega))^{1-\frac{1}{q}}} = \frac{||f||_p}{(vol(\Omega))^{\frac{1}{p}}}$$

Corollary 0.217. *Let f be holomorphic on Ω, then*

$$|f(z)| \leq c_n\frac{||f||_p}{(dist(z,b\Omega))^{\frac{2n}{p}}} \text{ and } ||f||_{L^\infty(K)} \leq c_{K,p}||f||_{L^p(\Omega)}, \forall K \subset \Omega \text{ compact}$$

Corollary 0.218. *Let $\{f_j\}$ be a sequence of holomorphic functions on Ω with $||f_j||_p$ bounded $\forall j$. Then there exists a subsequence convergent almost uniformly to a holomorphic function f on Ω.*

Suppose $f = \sum f_j(z)d\overline{z}_j$ is a (0,1)-form. What is the "size" of f at z_0?
An obvious choice would be: $\sqrt{\sum|f_j(z_0)|^2}$, but this does not transform well under a change in coordinates. Instead, use a strongly plurisubharmonic function ψ on Ω. For $v \in T_{z_0}\mathbb{C}^n$, let

$$||v||_{\psi,z_0} \equiv \psi''_{\mathbb{C}}(z_0,v) = \sum \psi_{j,\overline{k}}(z_0)v_j\overline{v_k} = \partial\overline{\partial}\psi(v,Jv).$$

$$||f||_{\psi,z_0} =^{\text{Def.}} \max\{|f\cdot v| \mid ||v||_{\psi,z_0} = 1\} = \sqrt{\sum \psi^{j,\overline{k}}(z_0)f_j(z_0)\overline{f_k(z_0)}},$$

where $\psi^{j,\overline{k}} = (\psi_{j,\overline{k}})^{-1}$.

Exercise 0.219. Let f be a (0,1)-form on Ω_2 and $\Omega \xrightarrow{\varphi} \Omega_2 \xrightarrow{\psi} \mathbb{R}$, where φ is biholomorphic and ψ is strongly plurisubharmonic, then $||f||_\psi \circ \varphi = ||\varphi^* f||_{\psi\circ\varphi}$.

Theorem 0.220 (Hormander's, V.1)**.** *Given:*

- *a strongly plurisubharmonic function ψ on a pseudoconvex domain Ω,*
- *a (0,1)-form f on Ω with $\overline{\partial}f = 0$, and*
- $\int_\Omega ||f||^2_\psi e^{-\psi}dV < \infty$,

then one can solve $\overline{\partial}u = f$ and $\int_\Omega |u|^2e^{-\psi}dV \leq \int_\Omega ||f||^2_\psi e^{-\psi}dV$.

Theorem 0.221 (Hormanders, V.2)**.** *Given:*

- *a strongly plurisubharmonic function ψ on a pseudoconvex domain Ω,*
- *ρ plurisubharmonic exhaustion function on Ω,*
- *f a $\overline{\partial}$-closed (0,1)-form on Ω, and*
- $\int ||f||^2_\psi e^{-\theta}dV < \infty$, *where* $\theta = \psi + \rho$,

then one can solve $\overline{\partial}u = f$ and $\int_\Omega |u|^2e^{-\theta}dV \leq \int_\Omega ||f||^2_\psi e^{-\theta}dV$.

Corollary 0.222. *Can always solve $\overline{\partial}u = f$ for any $\overline{\partial}$-closed (0,1)-form on a pseudoconvex domain.*

Corollary 0.223 (Oka-Weil Approximation Theorem)**.** *If $K = \widehat{K_{PLSH(\Omega)}} \subset \Omega \subset \mathbb{C}^n$, where K is compact and Ω is a pseudoconvex domain, and h is holomorphic on a neighborhood of K. Then h is a K-uniform limit of functions holomorphic on Ω (i.e. there is a sequence of functions on Ω which converge uniformly to h on K).*

Corollary 0.224. *Let K be a compact subset of a pseudoconvex domain Ω, then $\widehat{K_{Holo(\Omega)}} = \widehat{K_{PLSH(\Omega)}}$.*

Corollary 0.225. *Ω is a pseudoconvex domain $\Leftrightarrow \Omega$ is a plurisubharmonic convex hull $\Rightarrow \Omega$ is holomorphic convex hull $\Leftrightarrow^{Def.}$ Ω is a domain of holomorphy.*

Remark 0.226. A good online monograph for Hormander's theorem is by Berndtsson. This is what we would do next if the semester were longer.

Lecture 39. April 17, 2009

Lemma 0.227. *If $K \subset \Omega \subset \mathbb{C}^n$, where K compact and Ω open, and $z_0 \in \Omega$, then $\widehat{K \cup \{z_0\}}_{PLSH(\Omega)} = \hat{K}_{PLSH(\Omega)} \cup \{z_0\}$.*

Proof.
($\supseteq$) Trivial
($\subseteq$) $w \notin \hat{K} \cup \{z_0\} \Rightarrow \exists u$ plurisubharmonic on Ω such that $u(w) > max_K u$. Let $v = max\{u + \epsilon \log ||z - z_0||, -M\}$ for some fixed M. This is plurisubharmonic since if $f = (f_1, \ldots, f_l)$ is holomorphic $\Rightarrow \log ||f||$ plurisubharmonic. For ϵ small, $v(w) > max_{K \cup \{z_0\}}$ □

Lemma 0.228. *Given $K = \hat{K}_{PLSH(\Omega)} \subset U \subset^{open} \Omega \subset \mathbb{C}^n$, where Ω is pseudoconvex, then $\exists \rho$ plurisubharmonic on Ω with $\rho < 0$ on K, $\rho < 1$ on $\Omega \setminus u$.*

Proof. For $z \in \Omega \setminus K, \exists \psi_z$ continuous plurisubharmonic on Ω with $\psi_z < 0$ on K and $\psi_z > 1$ on a neighborhood of u_z of z. Also, $\exists \varphi$ plurisubharmonic exhaustion function on Ω, $\varphi < 0$ on K. $\{\varphi \leq 2\} \setminus U$ is compact so $\exists U_{z_1}, \ldots, U_{z_m}$, where each U_{z_j} is associated to a point z_j and together they cover $\{\varphi \leq 2\} \setminus U$. $\rho = max\{\psi_{z_1}, \ldots, \psi_{z_m}, \varphi\}$ works. □

Oka-Weil Approxiimation Theorem:
If $K = \hat{K}_{PLSH(\Omega)} \subset \Omega \subset \mathbb{C}^n$, where Ω is pseudoconvex, then every function holomorphic on a neighborhood of K is K-uniform limit of functions holomorphic on Ω.

Definition 0.229. K is **Runge** in Ω if every function holomorphic on a neighborhood of K is K-uniform limit of functions holomorphic on Ω.

Proof. (Assuming Hormander) Choose $\eta \in C_0^\infty(\Omega)$, $\eta = h$ on a bounded neighborhood U of K. Approximating $f = \eta + u$, need $\overline{\partial} u = -\overline{\partial}\eta$ for u small on K. Use Hormander (V.2) with $\psi = ||z||^2$ and $\rho = N\rho$ (by Hormander and the previous lemma). Solve $\overline{\partial} u = -\overline{\partial}\eta$ with

$$e^{-max_U ||z||^2} \int_{\{\rho<0\}} |u|^2 dV \int_\Omega |u|^2 e^{-||z||^2 - N\rho} dV \leq \int_\Omega ||\overline{\partial}\eta||^2_{Eucl.} e^{-||z||^2 - N\rho} dV \xrightarrow{N\to\infty} 0$$

By the solid mean value theorem and Cauchy-L^2 estimates,

$$||u||^2_{L^\infty(K)} = \text{constant} \leq e^{-max_U ||z||^2} \int_{\{\rho<0\}} |u|^2 dV.$$

□

Theorem 0.230. *$K = \hat{K}_{PLSH(\Omega)} \subset \Omega$ pseudconvex domain, then $K = \hat{K}_{Holo(\Omega)}$.*

Proof.
($\subset$) Trivial
($\supset$) Consider $z_0 \notin K$. Then $\widehat{K \cup \{z_0\}}_{PLSH} = \hat{K}_{PLSH} \cup \{z_0\} = K \cup \{z_0\}$. Pick $h = \begin{cases} 1 & \text{near } z_0 \\ 0 & \text{near } K \end{cases}$. By Oka-Weil, h is a $(K \cup \{z_0\})$-uniform limit of functions holomorphic on Ω. Therefore $z_0 \notin \hat{K}_{Holo(\Omega)}$. □

Corollary 0.231. *Let K be a compact subset of a pseudoconvex domain Ω. Then $\hat{K}_{PLSH} = \hat{K}_{Holo}$.*

Proof. $\hat{K}_{PLSH} \subset^{\text{always}} \hat{K}_{Holo} \subset^{\text{since }\Omega}_{\text{pseudoconvex}} (\widehat{\hat{K}_{PLSH}})_{Holo} = \hat{K}_{PLSH}$. □

Lecture 40. April 20, 2009

Outline of Proof for Hormander's Theorem
Case: $n = 1$, Version 1
For $f \in C(\Omega), u \in C^1(\Omega)$:

$$\frac{\partial u}{\partial \overline{z}} = f \Leftrightarrow \int_\Omega \frac{\partial u}{\partial \overline{z}} \alpha = \int_\Omega f\alpha, \forall \alpha \in C_0^\infty(\Omega)$$
$$\Leftrightarrow - \int_\Omega u \frac{\partial \alpha}{\partial \overline{z}} = \int_\Omega f\alpha, \forall \alpha \in C_0^\infty \dagger$$

If $u, f \in L^1_{loc}(\Omega)$ and † holds, say $\overline{\partial} u = f$ **weakly** (or "in the sense of distributions").

Proposition 0.232. *$\overline{\partial} u = 0$ weakly $\Rightarrow$ u agrees almost everywhere with a holomorphic function.*

Outline of Proof. On a slightly smaller set, we get a convolution approximation by $u_j \in C^\infty(\Omega)$ with $u_j \to u$ under the L^1-norm. By Fubini's theorem, $\overline{\partial} u_j = 0$ weakly, so u_j are holomorphic. Therefore $u_j \to u$ almost uniformly, and so u is holomorphic on Ω. □

Corollary 0.233. *Suppose f is C^1 and $\overline{\partial} u = f$ weakly, then u is equivalent to a C^1 function and $\overline{\partial} u = f$ classically.*

Proof. (Outline) On a slightly smaller set, use the Cauchy transform to get $v \in C^1$ with $\overline{\partial} v = f$ classically.

$$\overline{\partial}(u - v) = 0 \text{ weakly } \Rightarrow u - v \text{ holomorphic } \Rightarrow u = v + (u - v) \in C^1, \overline{\partial} u = \overline{\partial} v = f$$

□

Goal: Solve $\frac{\partial u}{\partial \overline{z}} = f$ with $\int |u|^2 e^{-\varphi} \leq \int \frac{|f|^2}{\varphi_{z\overline{z}}} e^{-\varphi}$.
$\exists u \in L^2(e^{-\varphi}) = \{v \mid \int |v|^2 e^{-\varphi} < \infty\}$ such that $\overline{\partial} u = f$ weakly.
Equivalently, $\exists u \in L^2(e^{-\varphi})$ such that $- \int u \frac{\partial \alpha}{\partial \overline{z}} = \int f\alpha, \forall \alpha \in C_0^\infty$.
Equivalently, (replacing α by $\overline{\alpha} e^{-\varphi}$) $\exists u \in L^2(e^{-\varphi})$ such that $- \int u(e^{\varphi} \frac{\partial}{\partial \overline{z}}(e^{-\varphi}\overline{\alpha}))e^{-\varphi} = \int f\overline{\alpha} e^{-\varphi}, \forall \alpha \in C_0^\infty(\Omega)$.
$- \int u(e^{\varphi} \frac{\partial}{\partial \overline{z}}(e^{-\varphi}\overline{\alpha}))e^{-\varphi} =$
$- \int u\overline{e^{\varphi} \frac{\partial}{\partial z}(e^{-\varphi}\alpha)}e^{-\varphi} \equiv \int u\overline{\overline{\partial^*}\varphi\alpha} e^{-\varphi}$
Notes

1. $\int u\overline{\overline{\partial^*}\varphi\alpha} e^{-\varphi} = \int (\overline{\partial} u)\overline{u} e^{-\varphi}$
2. $\overline{\partial^*}\varphi\alpha = 0, \alpha \in C_0^\infty \Rightarrow e^{-\varphi}\overline{\alpha}$ holomorphic $\Rightarrow \alpha = 0$.

Recall: Hahn-Banach and Riesz
$E \subset H$ is a subspace of a Hilbert space. Let $E \xrightarrow{T} \mathbb{C}$ be linear. Then $\exists u \in H$ such that $Tv = < v, u > \Leftrightarrow \exists C$ such that $|Tv| \leq C||v||, \forall v \in E$.
Apply with $E = \overline{\partial^*}\varphi(C_0^\infty)$ and $T = (\overline{\partial^*}\varphi\alpha) = \int \overline{f}\alpha e^{-\varphi}$.
Applying these result to our previous work we have:
$- \int u\overline{e^{\varphi} \frac{\partial}{\partial z}(e^{-\varphi}\alpha)}e^{-\varphi} \equiv \int u\overline{\overline{\partial^*}\varphi\alpha} e^{-\varphi} \Leftrightarrow \exists C$ such that $|\int f\overline{\alpha} e^{-\varphi}| \leq C\sqrt{\int |\overline{\partial^*}\varphi\alpha|^2 e^{-\varphi}}$

Want $\forall f \in L^2(\frac{e^{-\varphi}}{\varphi_{z\overline{z}}}), \exists u$ such that

$$\overline{\partial} u = f, ||u||_{L^2(e^{-\varphi})} \leq ||f||_{L^2(\frac{e^{-\varphi}}{\varphi_{z\overline{z}}})} \quad \Leftrightarrow$$

$$\forall f \in L^2(\frac{e^{-\varphi}}{\varphi_{z\overline{z}}}), |\int f\overline{\alpha} e^{-\varphi}| \leq ||f||_{L^2(\frac{e^{-\varphi}}{\varphi_{z\overline{z}}})} \quad \Leftrightarrow$$

$$\forall \alpha \in C_0^\infty, \int \varphi_{z\bar{z}}|\alpha|^2 e^{-\varphi} \leq ||\overline{\partial^*}\varphi\alpha||_{L^2(e^{-\varphi})}$$

So our previous goal, can be restated (from the above work) as:

$$\forall \alpha \in C_0^\infty, \int \varphi_{z\bar{z}}|\alpha|^2 e^{-\varphi} \leq ||\overline{\partial^*}\varphi\alpha||_{L^2(e^{-\varphi})}$$

$$\begin{aligned}\int |\overline{\partial^*}\varphi\alpha|^2 e^{-\varphi} &= \int (\overline{\partial}\overline{\partial^*}\varphi\alpha)\overline{\alpha}e^{-\varphi}\\ &= \int (\overline{\partial^*}\overline{\partial}\alpha)\overline{\alpha}e^{-\varphi} + \int \varphi_{z\bar{z}}|\alpha|^2 e^{-\varphi}\\ &= \int |\overline{\alpha}\alpha|^2 e^{-\varphi} + \int \varphi_{z\bar{z}}|\alpha| e^{-\varphi}\\ &\geq \int \varphi_{z\bar{z}}|\alpha| e^{-\varphi}\end{aligned}$$

In the second line we used: $\overline{\partial^*} = -\partial + \frac{\partial\varphi}{\partial\bar{z}}$. We also used: $\int \overline{\partial^*}\overline{\partial}\alpha\overline{\alpha}e^{-\varphi} = \int \overline{\partial}\alpha\overline{\overline{\partial}\alpha}e^{-\varphi}$

Lecture 41. April 20, 2009

What happens to the proof of Hormander's theorem when $n \geq 2$?
One approach:

- Work on strongly pseudoconvex domains Ω with smooth boundary (get general pseudoconvex Ω by passing to a limit)
- We have fewer "f" 's than when $n = 1$ - need $\overline{\partial}f = 0$
- This leads to a bigger class of test functions α:

$$\alpha, C^\infty \text{ vector fields on } \overline{\Omega}$$

$$\alpha(p) \in H_p(b\Omega), \forall p \in b\Omega$$

- Get a boundary term when integrating by parts in the previous calculation. In the case where $n = 1$, this was not a problems since the functions vanished on the boundary.
- The boundary term is: $\int_{b\Omega} \frac{\mathcal{L}(\alpha)}{\text{unit norm}} e^{-\varphi} dS \geq 0$ *(in several variables we need to assume that Ω is pseudoconvex)*

Related Results:

Theorem 0.234 (Ohsawa-Takegoshi Ext. Theorem (1987))**.** *Given :*

- *a pseudoconvex domain inside the unit ball, i.e. $\Omega \subset B^n$,*
- *a plurisubharmonic function φ on Ω,*
- *$E \subset \mathbb{C}^n$ a $\mathbb{C}$-affine subspace,*
- *f holomorphic on $\Omega \cap E = \emptyset$*
- $\int_{\Omega\cap E} |f|^2 e^{-\varphi} < \infty$

Then $\exists F$ holomorphic on Ω such that

$$F|_{\Omega\cap E} = f \text{ and}$$

$$\int_\Omega |F|^2 e^{-\varphi} \leq (4\pi)^{n-dim.E} \int_{\Omega\cap E} |f|^2 e^{-\varphi}$$

For the following assume that φ plurisubharmonic on the pseudconvex domain $\Omega \subset B^n$. Let

$$K_m(z) = sup\{|f(z)|^2 \mid f \in Holo(\Omega), \int_\Omega |f|^2 e^{-m\varphi} \leq 1\}$$

The "sup" can be replaced by "max."

Corollary 0.235. $(K_m(z))^{\frac{1}{m}} \xrightarrow{m\to\infty} e^{\varphi(z)}$ *almost uniformly if φ is continuous.*

Exercise 0.236. $K_m(z) = \sum_j |f_{j,m}(z)|^2$, where $\{f_{j,m}\}_{j=1}^{\infty}$ is an orthonormal basis for $L^2(\Omega, e^{-m\varphi}) \cap Holo(\Omega)$ *(weighted Bergman space).*

Corollary 0.237. $\varphi \approx \frac{\log(|f_{1,m}|^2+|f_{2,m}|^2+\dots)}{m}$ *(approximate this by a finite sum).*

Remark 0.238. Let $K(z,w) = \sum_j f_j(z)\overline{f_j(w)}$. Then $\int_\Omega h(w)K(z,w)e^{-m\varphi} = h(z)$, where h is holomorphic. This can be used to help solve the previous exercise.

Another important family of L^2 spaces in complex analysis:
Let $\Omega \subset \mathbb{C}^n$ be bounded with smooth (or "not too bad") boundary.
Hardy Space: $H(b\Omega) = L^2(b\Omega) \cap CR(b\Omega)$.
What measure is used for L^2 here?
When $n = 1$, arc length is a good measure to use.
When $n \geq 2$, most people use surface area in Euclidean space. However, this is not a particularly good choice because a change in coordinates has drastic effects on the integral. Other choices are more appropriate for specific problems.

Given $g \in H(b\Omega)$ we get an entire function Lg via the Laplace transform:

$$Lg(z) = \int_{w\in b\Omega} e^{zw}\overline{g(w)}d\sigma_w$$

(Part of) Paley-Weiner Theory: identify $L(H(b\Omega))$ as a weighted L^2-space of entire functions. Reasonable versions are avaliable for str. $\mathbb{C}$-linearly convex Ω (with smooth boundary). But they can be improved and extended. Using this (and other ideas) one can reduce the study of constant coefficient linear PDE on Ω to:

- multiplication of entire functions by polynomials
- dual $\mathbb{C}$-linear convex domains

These topics form part of the agenda for Math 703.